

On the noncommutative spin geometry of the standard Podleś sphere and index computations

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Abstract

The purpose of the paper is twofold: First, known results of the noncommutative spin geometry of the standard Podleś sphere are extended by discussing Poincaré duality and orientability. In the discussion of orientability, Hochschild homology is replaced by a twisted version which avoids the dimension drop. The twisted Hochschild cycle representing an orientation is related to the volume form of the distinguished covariant differential calculus. Integration over the volume form defines a twisted cyclic 2-cocycle which computes the q -winding numbers of quantum line bundles.

Second, a “twisted” Chern character from equivariant K_0 -theory to even twisted cyclic homology is introduced which gives rise to a Chern-Connes pairing between equivariant K_0 -theory and twisted cyclic cohomology. The Chern-Connes pairing between the equivariant K_0 -group of the standard Podleś sphere and the generators of twisted cyclic cohomology relative to the modular automorphism and its inverse are computed. This includes the pairings with the twisted cyclic 2-cocycle associated to the volume form, and the one corresponding to the “no-dimension drop” case. From explicit index computations, it follows that the pairings with these cocycles give the q -indices of the known equivariant 0-summable Dirac operator on the standard Podleś sphere.

Key words and phrases: Noncommutative geometry, K-theory, Chern character, index pairing, spectral triple, quantum spheres

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1 Introduction

As quantum groups and their associated quantum spaces describe geometric objects by noncommutative algebras, it is only natural to study them from Alain Connes' noncommutative geometry point of view [C1]. The first attempts were made in the nineties of the past century and exhibited some unexpected features. For instance, Masuda et al. noticed that the Hochschild dimension of the Podleś 2-spheres drops from the classical dimension 2 to 1 [MNW]. In another approach, Schmüdgen proved that some well-known covariant differential calculi on the quantum $SU(2)$ cannot be described by a Dirac operator [Sch].

These observations lowered the expectations on q -deformed spaces to be convincing examples of Connes' noncommutative geometry. The situation improved after the turn of the century when the first spectral triples on q -deformed spaces were constructed. Now there is a lively research activity in studying spectral triples on quantum groups and their associated quantum spaces. The best known examples are the (isospectral) spectral triples on the quantum $SU(2)$ [CP, DLSSV1] and the 0-dimensional (i.e., eigenvalues of exponential growth) spectral triple on the standard Podleś sphere [DS] with subsequent analysis of local index formulas in [C4, DLSSV2] and [NT2], respectively. Despite these positive results, some problems remained. For instance, an equivariant real structure was obtained in [DLSSV1] only after weakening the original conditions in [C2], and the 0-dimensional spectral triple in [DS] is not regular. To deal with such problems, it was repeatedly suggested to modify the original axioms of noncommutative spin geometry given in [C3].

Apart from merely providing examples, quantum group theory should be combined with Connes' noncommutative geometry. The basic input is equivariance [Sit], a property which is shared by all the above mentioned spectral triples. Another substantial step was made by Krähmer [Kr1] who constructed Dirac operators on quantum flag manifolds and proved that the Dirac operator defines a finite-dimensional covariant differential calculus in the sense of Woronowicz [Wor]. In the case of the standard Podleś sphere, considered as $\mathbb{C}P^1$, Krähmer's construction reproduces the spectral triple described by Dabrowski and Sitarz [DS]. Moreover, Kustermans et al. [KMT] developed a twisted version of cyclic cohomology in order to deal with the absence of graded traces on quantum groups and Hadfield [Had] showed that the dimension drop can be avoided for the Podleś spheres by considering the twisted Hochschild (co)homology.

Among all the examples mentioned so far, the standard Podleś sphere is distinguished. First of all, because it admits a real structure satisfying the

original conditions in [C2], and second, the Dirac operator fits nicely into Woronowicz’s theory of covariant differential calculi [Wor]. Furthermore, there is a twisted cyclic 2-cocycle associated to the volume form of the covariant differential calculus [SW2] which reappears in a local index formula and computes the quantum indices of the Dirac operator [NT2]. However, from Connes’ seven axioms in [C3], so far only four have been touched.

The main motivation behind the present paper is to expand the picture of noncommutative spin geometry of the standard Podleś sphere. Throughout the paper, we will work with the spectral triple found by Dabrowski and Sitarz [DS]. As indicated above, we partly have to modify the definitions in [C3]. The guideline is that the new structures should still allow the computation of the Chern-Connes and index pairings.

The Chern-Connes pairing will be defined between twisted cyclic cohomology and equivariant K_0 -theory. For this, we recall the definition of equivariant K_0 -theory in [NT1] and adapt it to our purpose. This means that we rephrase their definitions for the left-hand counterpart and take $*$ -structures into account. By relating equivariant K_0 -classes to Hilbert space representations of crossed product algebras, the equivariant K_0 -group of the standard Podleś sphere is easily obtained from the results in [SW3]. More precisely, we show that it is freely generated by the (equivalence classes of) quantum line bundles of each winding number.

For the Chern-Connes pairing, we need a “twisted” Chern character mapping equivariant K_0 -classes into twisted cyclic homology. Section 4 introduces a twisted Chern character in a general setting. The only requirements are an appropriate notion of equivariance and a compatibility condition on the twisting automorphism.

The modular automorphism associated to the Haar state on quantum $SU(2)$ restricts to an automorphism of the standard Podleś sphere, and the volume form defines a twisted cyclic 2-cocycle relative to it. In Section 5.1, we compute the full pairing between equivariant K_0 -theory and twisted cyclic cohomology with respect to this automorphism. However, this does not correspond to the “no-dimension drop” case of twisted cyclic cohomology. The dimension drop can be avoided by considering the inverse modular automorphism. A corresponding twisted cyclic 2-cocycle, which is also non-trivial on Hochschild homology, was found by Krähmer [Kr2]. The Chern-Connes pairing between equivariant K_0 -theory and this twisted cyclic 2-cocycle is computed in Section 5.2. Notably, both twisted cyclic 2-cocycles compute the q -winding numbers.

The discussion of the orientability in Section 6 demonstrates that, in our example, the twisted versions of Hochschild and cyclic (co)homology fit much better into the framework: First, there is a twisted Hochschild 2-cycle such

that its Hilbert space representation by taking commutators with the Dirac operator gives the q -grading operator, so the spectral triple satisfies a modified orientability axiom. Second, the twisted 2-cycle defines a non-trivial class in the twisted Hochschild homology and also in the twisted cyclic homology. In particular, it corresponds to the “no dimension drop” case. Third, the representation of the twisted 2-cycle as a 2-form of the algebraically defined covariant differential calculus yields the unique (up to a constant) volume form. And finally, integration over this volume form defines a twisted cyclic 2-cocycle which computes q -indices of the Dirac operator. Note that the combination of the first and third remark bridges nicely Connes’ and Woronowicz’ notion of a volume form. It would be interesting to see whether similar results can be obtained for other q -deformed spaces.

The spectral triple under discussion is not regular. Regularity provides an operatorial formulation of the calculus of smooth functions needed in the local index formula. We do not insist on regularity as long as index computations are possible. In our example, the indices can be computed by elementary methods using predominantly equivariance. In Section 7, the index and q -index of the Dirac operator paired with any K_0 -class are calculated. As expected, we get the winding number (an integer) in the first case and the q -winding number (a q -integer) in the second. Moreover, it is shown that Poincaré duality—one of Connes’ seven axioms—holds.

Finiteness, another axiom, is implicitly fulfilled by the definition of the spinor space as projective modules in Section 2.3. We refrain from the technical details of extending the coordinate algebra to obtain a pre- C^* -algebra.

Although this paper deals only with the standard Podleś sphere, our approach to the Chern-Connes pairing between equivariant K_0 -theory and twisted cyclic cohomology is general enough to be applicable to other examples. For instance, Remark 4.5 yields a pairing of twisted cyclic cohomology with equivariant K_0 -theory, where the only equivariance condition is the compatibility of the twisting automorphism with the involution of the algebra. In order not to overstretch the scope of the paper, we did not consider K_1 -theory. A definition of a modified K_1 -group which uses the modular automorphism can be found in [CPR].

In this regard, let us remark that our definition of an equivariant K_0 -group does not only apply to algebras with a modular automorphism. Of course, changing the automorphism might change the equivariant K_0 -group as much as it might change the twisted cyclic (co)homology. In the best cases, it will be isomorphic to the original one, as it happens for the inverse modular automorphism in the present paper.

2 Preliminaries

2.1 Crossed product algebras

Throughout the paper, we will always work over the complex numbers \mathbb{C} . Let \mathcal{U} be a Hopf $*$ -algebra and \mathcal{B} a left \mathcal{U} -module $*$ -algebra, that is, \mathcal{B} is a unital $*$ -algebra with left \mathcal{U} -action \triangleright satisfying

$$f \triangleright xy = (f_{(1)} \triangleright x)(f_{(2)} \triangleright y), \quad f \triangleright 1 = \varepsilon(f)1, \quad (f \triangleright x)^* = S(f)^* \triangleright x^* \quad (1)$$

for $x, y \in \mathcal{B}$ and $f \in \mathcal{U}$. Here and throughout the paper, ε denotes the counit, S the antipode, and $\Delta(f) = f_{(1)} \otimes f_{(2)}$, $f \in \mathcal{U}$, is the Sweedler notation for the comultiplication.

The *left crossed product $*$ -algebra* $\mathcal{B} \rtimes \mathcal{U}$ is defined as the $*$ -algebra generated by the two $*$ -subalgebras \mathcal{B} and \mathcal{U} with respect to the crossed commutation relations

$$fx = (f_{(1)} \triangleright x)f_{(2)}, \quad x \in \mathcal{B}, \quad f \in \mathcal{U}. \quad (2)$$

Suppose there exists a faithful state h on \mathcal{B} which is \mathcal{U} -invariant, i.e.,

$$h(f \triangleright x) = \varepsilon(f)h(x), \quad x \in \mathcal{B}, \quad f \in \mathcal{U}.$$

Then there is a unique $*$ -representation π_h of $\mathcal{B} \rtimes \mathcal{U}$ on the domain \mathcal{B} with inner product $\langle x, y \rangle := h(x^*y)$ such that

$$\pi_h(y)x = yx, \quad \pi_h(f)x = f \triangleright x, \quad x, y \in \mathcal{B}, \quad f \in \mathcal{U}. \quad (3)$$

These left-handed definitions have right-handed counterparts. The right \mathcal{U} -action on a right \mathcal{U} -module $*$ -algebra satisfies

$$xy \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)}), \quad 1 \triangleleft f = \varepsilon(f)1, \quad (x \triangleleft f)^* = x^* \triangleleft S(f)^*, \quad (4)$$

the crossed commutation relations of the *right crossed product $*$ -algebra* $\mathcal{U} \ltimes \mathcal{B}$ read

$$xf = f_{(1)}(x \triangleleft f_{(2)}), \quad x \in \mathcal{B}, \quad f \in \mathcal{U}, \quad (5)$$

the invariant state fulfills

$$h(x \triangleleft f) = \varepsilon(f)h(x), \quad x \in \mathcal{B}, \quad f \in \mathcal{U},$$

and the $*$ -representation π_h on \mathcal{B} is given by

$$\pi_h(y)x = yx, \quad \pi_h(f)x = x \triangleleft S^{-1}(f), \quad x, y \in \mathcal{B}, \quad f \in \mathcal{U}. \quad (6)$$

Proofs of these facts can be found in [SW1] and [SW3].

2.2 Hopf fibration of quantum $SU(2)$

Throughout this paper, q stands for a positive real number such that $q \neq 1$, and we set $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$, where $x \in \mathbb{R}$. For more details on the algebras introduced in this section, we refer to [KS].

The Hopf $*$ -algebra $\mathcal{U}_q(\mathfrak{su}_2)$ has four generators E, F, K, K^{-1} with defining relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quad FK = qKF, \quad EF - FE = \frac{1}{q - q^{-1}}(K^2 - K^{-2}),$$

involution $E^* = F, K^* = K$, comultiplication

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K,$$

counit $\varepsilon(E) = \varepsilon(F) = \varepsilon(K - 1) = 0$, and antipode $S(K) = K^{-1}, S(E) = -qE, S(F) = -q^{-1}F$.

The coordinate Hopf $*$ -algebra of quantum $SU(2)$ will be denoted by $\mathcal{O}(SU_q(2))$. A definition of $\mathcal{O}(SU_q(2))$ in terms of generators and relations can be found in [KS]. Recall from the Peter-Weyl theorem for compact quantum groups that a linear basis of $\mathcal{O}(SU_q(2))$ is given by the matrix elements t_{jk}^l of finite dimensional unitary corepresentations, where $l \in \frac{1}{2}\mathbb{N}_0$ and $j, k = -l, -l+1, \dots, l$. These matrix elements satisfy

$$\Delta(t_{jk}^l) = \sum_{n=-l}^l t_{jn}^l \otimes t_{nk}^l, \quad \varepsilon(t_{jk}^l) = \delta_{jk}, \quad S(t_{jk}^l) = t_{kj}^{l*}, \quad (7)$$

where δ_{jk} stands for the Kronecker delta. It follows immediately that

$$\sum_{n=-l}^l t_{nj}^{l*} t_{nk}^l = \sum_{n=-l}^l t_{jn}^l t_{kn}^{l*} = \delta_{jk}. \quad (8)$$

The standard generators of $\mathcal{O}(SU_q(2))$, usually denoted by a and c , are given by $a = t_{-1/2, -1/2}^{1/2}$ and $c = t_{1/2, -1/2}^{1/2}$. An explicit description of t_{nk}^l in terms of the generators of $\mathcal{O}(SU_q(2))$ can be found in [KS, Section 4.2.4].

The Haar state h on $\mathcal{O}(SU_q(2))$ is given by $h(t_{00}^0) = 1$ and $h(t_{jk}^l) = 0$ for $l > 0$. Since h is faithful, we can define an inner product on $\mathcal{O}(SU_q(2))$ by $\langle x, y \rangle := h(x^*y)$. With respect to this inner product, the elements

$$v_{jk}^l := [2l+1]_q^{1/2} q^j t_{jk}^l \quad (9)$$

form an orthonormal vector space basis of $\mathcal{O}(SU_q(2))$.

There is a left and a right $\mathcal{U}_q(\mathfrak{su}_2)$ -action on $\mathcal{O}(SU_q(2))$ turning it into a $\mathcal{U}_q(\mathfrak{su}_2)$ -module $*$ -algebra. Since h is $\mathcal{U}_q(\mathfrak{su}_2)$ -invariant, Equations (3) and (6) define $*$ -representations of $\mathcal{U}_q(\mathfrak{su}_2)$. To distinguish these representations, we shall omit the representation π_h in the first case and write ∂_f instead of $\pi_h(f)$

in the second case. On the basis vectors v_{jk}^l , the actions of the generators E , F and K read

$$E \triangleright v_{jk}^l = \alpha_k^l v_{j,k+1}^l, \quad F \triangleright v_{jk}^l = \alpha_{k-1}^l v_{j,k-1}^l, \quad K \triangleright v_{jk}^l = q^k v_{jk}^l, \quad (10)$$

$$\partial_E(v_{jk}^l) = -\alpha_{j-1}^l v_{j-1,k}^l, \quad \partial_F(v_{jk}^l) = -\alpha_j^l v_{j+1,k}^l, \quad \partial_K(v_{jk}^l) = q^{-j} v_{jk}^l, \quad (11)$$

where $\alpha_j^l := ([l-j]_q [l+j+1]_q)^{1/2}$.

Note that, by Equations (4) and (6),

$$\partial_X(ab) = \partial_{X(2)}(a) \partial_{X(1)}(b), \quad \partial_X(a^*) = \partial_{S(X^*)}(a)^* \quad (12)$$

for $a, b \in \mathcal{O}(\mathrm{SU}_q(2))$ and $X \in \mathcal{U}_q(\mathrm{su}_2)$. We use the right action ∂_{K^2} of the group-like element K^2 on $\mathcal{O}(\mathrm{SU}_q(2))$ to define a Hopf fibration. For $N \in \mathbb{Z}$, set

$$M_N := \{x \in \mathcal{O}(\mathrm{SU}_q(2)) : \partial_{K^2}(x) = q^{-N}x\}$$

and denote by \overline{M}_N its Hilbert space closure. Equation (11) implies

$$M_N = \mathrm{span}\{v_{N/2,k}^l : k = -l, \dots, l, \quad l = \frac{|N|}{2}, \frac{|N|}{2} + 1, \dots\}$$

We summarize some basic properties of M_N in the following lemma.

Lemma 2.1. *For $N, K \in \mathbb{Z}$ and $f \in \mathcal{U}_q(\mathrm{su}_2)$,*

$$M_N^* \subset M_{-N}, \quad M_N M_K \subset M_{N+K}, \quad \mathrm{span}\{a_N b_K \in M_N M_K\} = M_{N+K}, \quad (13)$$

$$\partial_E(M_N) \subset M_{N-2}, \quad \partial_F(M_N) \subset M_{N+2}, \quad (14)$$

$$f \triangleright M_N \subset M_N, \quad f \in \mathcal{U}_q(\mathrm{su}_2). \quad (15)$$

In particular, M_0 is a $$ -algebra and a left $\mathcal{U}_q(\mathrm{su}_2)$ -module $*$ -subalgebra of $\mathcal{O}(\mathrm{SU}_q(2))$, M_N is a M_0 -bimodule and a left $\mathcal{U}_q(\mathrm{su}_2)$ -module, and the restriction of the representation π_h from (3) to M_0 and $\mathcal{U}_q(\mathrm{su}_2)$ defines a $*$ -representation of $M_0 \rtimes \mathcal{U}_q(\mathrm{su}_2)$ on M_N . Moreover, as a left or right $\mathcal{O}(S_q^2)$ -module, M_N is generated by $v_{N,-N}^{|N|}, \dots, v_{N,N}^{|N|}$ defined in (9).*

Proof. Most of the assertions are easy consequences of Equations (1), (4), (10) and (11). The last relation in (13) follows from a Schur type argument since $\mathrm{span}\{a_N b_K \in M_N M_K\} \subset M_{N+K}$ is a left $M_0 \rtimes \mathcal{U}_q(\mathrm{su}_2)$ -module and M_{N+K} is an irreducible one [SW3]. The last claim can be proved by using an explicit description of the matrix coefficients t_{jk}^l in (9) (see, e.g., [KS]). \square

The $*$ -algebra M_0 is known as the standard Podleś sphere $\mathcal{O}(S_q^2)$. Usually one defines $\mathcal{O}(S_q^2)$ as the abstract unital $*$ -algebra with three generators B , B^* , $A = A^*$ and defining relations [Pod]

$$BA = q^2 AB, \quad AB^* = q^2 B^* A, \quad B^* B = A - A^2, \quad BB^* = q^2 A - q^4 A^2. \quad (16)$$

Setting

$$A = t_{1/2, -1/2}^{1/2*} t_{1/2, -1/2}^{1/2}, \quad B = t_{1/2, 1/2}^{1/2*} t_{1/2, -1/2}^{1/2}, \quad B^* = t_{1/2, -1/2}^{1/2*} t_{1/2, 1/2}^{1/2}, \quad (17)$$

yields an embedding into $\mathcal{O}(\mathrm{SU}_q(2))$ and an isomorphism between $\mathcal{O}(\mathrm{S}_q^2)$ and M_0 (see, e.g., [KS]). For generators, the crossed commutation relations (2) in $\mathcal{O}(\mathrm{S}_q^2) \rtimes \mathcal{U}_q(\mathrm{su}_2)$ can easily be obtained from (1) and (10).

Recall that an automorphism θ satisfying $\varphi(xy) = \varphi(\theta(y)x)$ for a state φ on a certain $*$ -algebra is called a modular automorphism (associated to φ). It can be shown that $h(xy) = h(\theta(y)x)$ for $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$, where

$$\theta(y) = \partial_{K^2}(K^{-2} \triangleright y) = K^{-2} \triangleright \partial_{K^2}(y). \quad (18)$$

The restriction of h to $\mathcal{O}(\mathrm{S}_q^2)$ defines a faithful invariant state on $\mathcal{O}(\mathrm{S}_q^2)$ with modular automorphism

$$\theta(y) = K^{-2} \triangleright y, \quad y \in \mathcal{O}(\mathrm{S}_q^2). \quad (19)$$

This follows from (18) and the ∂_{K^2} -invariance of $\mathcal{O}(\mathrm{S}_q^2)$. By the third relation in (1), the modular automorphism obeys $\theta(y)^* = \theta^{-1}(y^*)$ for all $y \in \mathcal{O}(\mathrm{S}_q^2)$. On the generators B , B^* and A , the action of θ is given by

$$\theta(B) = q^2 B, \quad \theta(B^*) = q^{-2} B^*, \quad \theta(A) = A.$$

2.3 Dirac operator on the standard Podleś sphere

On the standard Podleś sphere, there are two non-isomorphic spectral triples known: the 0-dimensional spectral triple described in [DS] and the isospectral one from [DDLW]. Both were found by explicit computations on a Hilbert space basis. However, the 0-dimensional spectral triple admits a convenient description by using an embedding of the quantum spinor bundle into the Hopf $*$ -algebra $\mathcal{O}(\mathrm{SU}_q(2))$ [SW2]. This construction is unique to the standard Podleś sphere; the general construction of Dirac operators on quantum flag manifolds in [Kr1] differs slightly from this.

Because of its relation to the representation theory of $\mathcal{O}(\mathrm{SU}_q(2))$, we will work in this paper only with the 0-dimensional spectral triple. The presentation below gives an overview of the results in [SW2] including simplified “coordinate free” proofs.

We define the quantum spinor bundle as the subspace

$$W := M_{-1} \oplus M_1 \subset \mathcal{O}(\mathrm{SU}_q(2))$$

with inner product $\langle x, y \rangle = h(x^*y)$, and set $\mathcal{H} := \overline{M}_{-1} \oplus \overline{M}_1$. The $*$ -representation of $\mathcal{O}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}_2)$ described in (3) restricts on W to a

representation of the crossed product algebra $\mathcal{O}(\mathbb{S}_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$. For simplicity of notation, we shall omit the symbol π_h of the representation.

By Lemma 2.1 and Equation (11), the operator

$$D_0 := \begin{pmatrix} 0 & \partial_E \\ \partial_F & 0 \end{pmatrix}$$

maps W into itself and $\{\frac{1}{\sqrt{2}}(v_{-1/2,k}^l, \pm v_{1/2,k}^l)^\dagger : k = -l, \dots, l, l = \frac{1}{2}, \frac{3}{2}, \dots\}$ forms a complete set of orthonormal eigenvectors. It follows that the closure of D_0 is a self-adjoint operator, called the Dirac operator D . The corresponding eigenvalues depend only on l and the sign \pm , and are given by $\pm[l + \frac{1}{2}]_q$. In particular, D has compact resolvent.

By (12), $\partial_X(xy) = \partial_X(x)\partial_{K^{-1}}(y) + \partial_K(x)\partial_X(y)$ for $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$ and $X = E, F$. Using $\partial_K(v_N) = q^{-N/2}v_N$ for $v_N \in M_N$, one computes

$$[D, a] = \begin{pmatrix} 0 & q^{1/2}\partial_E(a) \\ q^{-1/2}\partial_F(a) & 0 \end{pmatrix}, \quad a \in \mathcal{O}(\mathbb{S}_q^2). \quad (20)$$

As a consequence, $[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{O}(\mathbb{S}_q^2)$.

There is a natural grading operator γ on \mathcal{H} given by

$$\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, $D\gamma = -\gamma D$ and $\gamma a = a\gamma$ for all $a \in \mathcal{O}(\mathbb{S}_q^2)$.

Set $J_0 := * \circ \partial_K K^{-1}$. For $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$,

$$\begin{aligned} \langle J_0(x), J_0(y) \rangle &= h((\partial_K K^{-1} \triangleright x)(\partial_{K^{-1}} K \triangleright y^*)) = h((\partial_K K^{-1} \triangleright y^*)(\partial_K K^{-1} \triangleright x)) \\ &= h(\partial_K K^{-1} \triangleright (y^* x)) = h(y^* x) = \langle y, x \rangle \end{aligned}$$

by Equations (1), (4), (18), and the $\mathcal{U}_q(\mathfrak{su}_2)$ -invariance of h . Hence J_0 is anti-unitary. From $* \circ \partial_K K^{-1} = \partial_{K^{-1}} K \circ *$, we conclude $J_0^2 = 1$ and

$$J_0 a^* J_0^{-1} x = J_0 a^* J_0 x = \partial_{K^{-1}} K \triangleright (a^* (\partial_K K^{-1} \triangleright x)^*)^* = x(K \triangleright a) \quad (21)$$

for $a \in \mathcal{O}(\mathbb{S}_q^2)$ and $x \in \mathcal{O}(\mathrm{SU}_q(2))$. By Lemma 2.1, J_0 leaves W invariant. Since the representations of $\mathcal{O}(\mathbb{S}_q^2)$ and $[D, a]$, $a \in \mathcal{O}(\mathbb{S}_q^2)$, act on W by left multiplication, it follows from (21) that

$$[a, J_0 b^* J_0^{-1}] = 0, \quad [[D, a], J_0 b^* J_0^{-1}] = 0, \quad a, b \in \mathcal{O}(\mathbb{S}_q^2).$$

The last relations in (4) and (6) imply $J_0 \partial_f = \partial_{S(K^{-1} f^* K)} J_0$ and therefore $J_0 D = -D J_0$ on W . Clearly, $\gamma J_0 = -J_0 \gamma$ on W by (13). Considering $J := \gamma J_0$ as an anti-unitary operator on \mathcal{H} , one gets $J^2 = -1$ and $J D = D J$. Summing up, we obtain the following theorem [SW2, Theorem 3.3 (iii)].

Theorem 2.2. *The quintuple $(\mathcal{O}(S_q^2), \mathcal{H}, D, J, \gamma)$ is a real even spectral triple in the sense of [C2].*

The formulas $\Omega := \text{span}\{b[D, a] : a, b \in \mathcal{O}(S_q^2)\}$ and $da := i[D, a]$ define a covariant first order differential calculus $d : \mathcal{O}(S_q^2) \rightarrow \Omega$. The corresponding universal differential calculus is given by $\Omega^\wedge = \bigoplus_{k=0}^\infty \Omega^{\otimes k} / \mathcal{I}$, where $\Omega^{\otimes k} = \Omega \otimes_{\mathcal{O}(S_q^2)} \cdots \otimes_{\mathcal{O}(S_q^2)} \Omega$ (k -times), $\Omega^{\otimes 0} = \mathcal{O}(S_q^2)$, and \mathcal{I} denotes the two-sided ideal in the tensor algebra $\bigoplus_{k=0}^\infty \Omega^{\otimes k}$ generated by the elements $\sum_i da_i \otimes db_i$ such that $\sum_i a_i db_i = 0$. The product in the algebra Ω^\wedge is denoted by \wedge and we write $d(a_0 da_1 \wedge \cdots \wedge da_k) = da_0 \wedge da_1 \wedge \cdots \wedge da_k$. The following facts are proved in [SW2].

Proposition 2.3. *There exists an invariant 2-form $\omega \in \Omega^{\wedge 2}$ such that $\Omega^{\wedge 2}$ is the free $\mathcal{O}(S_q^2)$ -module generated by ω with $a\omega = \omega a$ for all $a \in \mathcal{O}(S_q^2)$ and $a_0 da_1 \wedge da_2 = a_0(q\partial_E(a_1)\partial_F(a_2) - q^{-1}\partial_F(a_1)\partial_E(a_2))\omega$. Let h denote the Haar state on $\mathcal{O}(S_q^2)$ and θ its modular automorphism described in (19). The linear functional $\int : \Omega^{\wedge 2} \rightarrow \mathbb{C}$, $\int a\omega := h(a)$ defines a non-trivial θ -twisted cyclic 2-cocycle τ on $\mathcal{O}(S_q^2)$ given by*

$$\tau(a_0, a_1, a_2) = \int a_0 da_1 \wedge da_2 = h\left(a_0(q\partial_E(a_1)\partial_F(a_2) - q^{-1}\partial_F(a_1)\partial_E(a_2))\right). \quad (22)$$

We call ω a volume form associated with the covariant differential calculus. For a definition of twisted cyclic cocycles, see Section 4. An explicit expressions of ω can easily be deduced from the formulas given in Section 6 and in [SW2, Appendix (Proof of Lemma 4.4)].

3 Equivariant K_0 -theory

3.1 Definition and basic material

This section is concerned with a simple definition of equivariant K_0 -theory. The approach follows closely the lines of [NT1] which works well for compact quantum groups. Our treatment differs from that in [NT1] in two aspects. First, we take $*$ -structures into account, and second, we will include also *left* crossed product algebras in our considerations.

We start by recalling some definitions from [NT1]. Let \mathcal{B} be a right \mathcal{U} -module algebra. Suppose that $\rho^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ is a finite dimensional representation of the opposite algebra \mathcal{U}° or, equivalently, $\rho^\circ : \mathcal{U} \rightarrow \text{End}(\mathbb{C}^n)$ is a finite dimensional *anti*-homomorphism. Then $\mathbb{C}^n \otimes \mathcal{B}$ inherits a right

$\mathcal{U} \ltimes \mathcal{B}$ -module structure from the anti-representation

$$\pi^\circ(f)(v \otimes a) := \rho^\circ(f_{(1)})v \otimes a \triangleleft f_{(2)}, \quad \pi^\circ(b)(v \otimes a) := v \otimes ab, \quad f \in \mathcal{U}, b \in \mathcal{B}. \quad (23)$$

The algebra $M_{n \times n}(\mathbb{C}) \otimes \mathcal{B}$ can be embedded into $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$ by

$$M_{n \times n}(\mathbb{C}) \otimes \mathcal{B} \ni T \otimes b \longmapsto T \otimes L_b \in \text{End}(\mathbb{C}^n \otimes \mathcal{B}),$$

where $L_b a := ba$, $a, b \in \mathcal{B}$, denotes the left multiplication of \mathcal{B} . On column vectors $\mathbf{b} \in \mathcal{B}^n \cong \mathbb{C}^n \otimes \mathcal{B}$, the action of $X \in M_{n \times n}(\mathcal{B}) \cong M_{n \times n}(\mathbb{C}) \otimes \mathcal{B}$ is conveniently expressed by matrix multiplication, i.e.,

$$M_{n \times n}(\mathcal{B}) \ni X \longmapsto (\mathbf{b} \mapsto X \mathbf{b}) \in \text{End}(\mathbb{C}^n \otimes \mathcal{B}). \quad (24)$$

We turn $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$ into a right \mathcal{U} -module by using the left adjoint action of \mathcal{U}° , i.e.

$$\text{ad}_L^\circ(f)(X) := \pi^\circ(f_{(1)})X\pi^\circ(S^{-1}(f_{(2)})), \quad X \in \text{End}(\mathbb{C}^n \otimes \mathcal{B}), \quad f \in \mathcal{U}.$$

From [NT1, Lemma 1.1] (or from direct calculations), it follows that

$$\text{ad}_L^\circ(f)(T \otimes L_b) = \rho^\circ(f_{(1)})T\rho^\circ(S^{-1}(f_{(3)})) \otimes L_{b \triangleleft f_{(2)}} \quad (25)$$

for all $T \otimes b \in M_{n \times n}(\mathbb{C}) \otimes \mathcal{B}$ and $f \in \mathcal{U}$. Under the identification (24), Equation (25) becomes

$$\text{ad}_L^\circ(f)(X) = \rho^\circ(f_{(1)})(X \triangleleft f_{(2)})\rho^\circ(S^{-1}(f_{(3)})), \quad (26)$$

where $X \in M_{n \times n}(\mathcal{B})$, $f \in \mathcal{U}$, and $X \triangleleft f$ stands for the action of f on each entry of the matrix X . Looking at the last equations, one readily sees that $M_{n \times n}(\mathcal{B})$ is a right \mathcal{U} -module subalgebra of $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$. Alternatively, we can consider $M_{n \times n}(\mathcal{B})$ as a left \mathcal{U}° -module subalgebra of $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$.

Now we take $*$ -structures into account. Suppose that \mathcal{U} is a Hopf $*$ -algebra, \mathcal{B} a right \mathcal{U} -module $*$ -algebra, and $\rho^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ a $*$ -representation. In order to turn $M_{n \times n}(\mathcal{B})$ into a \mathcal{U}° -module $*$ -algebra, we need an automorphism $\sigma : \mathcal{B} \rightarrow \mathcal{B}$ such that $\sigma(a \triangleleft f) = \sigma(a) \triangleleft S^{-2}(f)$ and $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{B}$ and $f \in \mathcal{U}$. Then $X^\dagger := \sigma(X)^*$ defines an involution on $M_{n \times n}(\mathcal{B})$ such that

$$\begin{aligned} (\text{ad}_L^\circ(f)(X))^\dagger &= \rho^\circ(S^{-1}(f_{(3)})^*)(\sigma(X)^* \triangleleft S^{-1}(f_{(2)})^*)\rho^\circ(f_{(1)}^*) \\ &= \text{ad}_L^\circ(S^{-1}(f)^*)(X^\dagger) \end{aligned}$$

for all $X \in M_{n \times n}(\mathcal{B})$ and $f \in \mathcal{U}$. As S^{-1} is the antipode of \mathcal{U}° , the last relation shows that $M_{n \times n}(\mathcal{B})$ with the involution † and the left \mathcal{U}° -action ad_L° is a left \mathcal{U}° -module $*$ -algebra.

A matrix $X \in M_{n \times n}(\mathcal{B})$ is called *(right) \mathcal{U} -invariant* if there exists a $*$ -representation $\rho^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ such that

$$\text{ad}_L^\circ(f)(X) = \varepsilon(f)X$$

for all $f \in \mathcal{U}$.

Next we make analogous definitions for left crossed product algebras. Let thus \mathcal{B} be a *left* \mathcal{U} -module algebra. In order to apply the definitions given above, we consider \mathcal{B} as a *right* \mathcal{U}^{cop} -module with right \mathcal{U}^{cop} -action given by $a \triangleleft f := S^{-1}(f) \triangleright a$, where $f \in \mathcal{U}$ and $a \in \mathcal{B}$. Then ad_L° turns $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$ into a left $\mathcal{U}^{\circ, \text{cop}}$ -module algebra. To get back to a \mathcal{U}° -module algebra, we use again the inverse of the antipode and define a right \mathcal{U}° -action on $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$ by setting $\text{ad}_R^\circ(f) := \text{ad}_L^\circ(S^{-1}(f))$. Now $M_{n \times n}(\mathcal{B})$ becomes a right \mathcal{U}° -module subalgebra and, for all $X \in M_{n \times n}(\mathcal{B})$,

$$\text{ad}_R^\circ(f)(X) = \rho^\circ(S^{-1}(f_{(1)}))(S^{-2}(f_{(2)}) \triangleright X) \rho^\circ(f_{(3)}). \quad (27)$$

Similarly to the above, we assume that there is an automorphism $\sigma : \mathcal{B} \rightarrow \mathcal{B}$ such that $\sigma(f \triangleright a) = S^2(f) \triangleright \sigma(a)$ and $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{B}$ and $f \in \mathcal{U}$. With respect to the involution $X^\dagger := \sigma(X)^*$, we get

$$\begin{aligned} (\text{ad}_R^\circ(f)(X))^\dagger &= \rho^\circ(f_{(3)}^*)(S(f_{(2)})^* \triangleright \sigma(X)^*) \rho^\circ(S^{-1}(f_{(1)})^*) \\ &= \text{ad}_R^\circ(S^{-1}(f)^*)(X^\dagger) \end{aligned}$$

for all $X \in M_{n \times n}(\mathcal{B})$ and $f \in \mathcal{U}$ since $S^{-1} \circ * = * \circ S$. Hence the involution † and the right \mathcal{U}° -action ad_R° endow $M_{n \times n}(\mathcal{B})$ with the structure of a right \mathcal{U}° -module $*$ -algebra. Note that the automorphism S^{-2} in Equation (27) is necessary for $(\text{ad}_R^\circ(f)(X))^\dagger = \text{ad}_R^\circ(S^{-1}(f)^*)(X^\dagger)$ to hold.

As above, we say that $X \in M_{n \times n}(\mathcal{B})$ is *(left) \mathcal{U} -invariant* if there exists a $*$ -representation $\rho^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ such that

$$\text{ad}_R^\circ(f)(X) = \varepsilon(f)X$$

for all $f \in \mathcal{U}$.

For a definition of equivariant K_0 -theory, we shall use the Murray-von Neumann equivalence of projections. Given an automorphism σ of \mathcal{B} such that $\sigma(b)^* = \sigma^{-1}(b^*)$, an idempotent $P \in M_{n \times n}(\mathcal{B})$ will be called *projection* if $P = P^\dagger$.

Definition 3.1. Let \mathcal{B} be a $*$ -algebra and $\sigma : \mathcal{B} \rightarrow \mathcal{B}$ an automorphism satisfying $\sigma(b)^* = \sigma^{-1}(b^*)$. Suppose that \mathcal{B} is a right \mathcal{U} -module $*$ -algebra and $\sigma(a \triangleleft f) = \sigma(a) \triangleleft S^{-2}(f)$ for all $a \in \mathcal{B}$ and $f \in \mathcal{U}$; or \mathcal{B} is a left \mathcal{U} -module $*$ -algebra and $\sigma(f \triangleright a) = S^2(f) \triangleright \sigma(a)$.

For $n, m \in \mathbb{N}$, let $\rho_1^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ and $\rho_2^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^m)$ be finite dimensional $*$ -representations. Denote by π_1° and π_2° the representations of \mathcal{U}° on $\mathbb{C}^n \otimes \mathcal{B}$ and $\mathbb{C}^m \otimes \mathcal{B}$, respectively, given in Equation (23) or (29). We say that invariant projections $P \in M_{n \times n}(\mathcal{B})$ and $Q \in M_{m \times m}(\mathcal{B})$ are *Murray-von Neumann equivalent* if there exists a $V \in \text{Hom}_{\mathcal{B}}(\mathbb{C}^n \otimes \mathcal{B}, \mathbb{C}^m \otimes \mathcal{B})$ such that $V^\dagger V = P$, $VV^\dagger = Q$ and $\pi_2(f)V = V\pi_1(f)$ for all $f \in \mathcal{U}$.

Remark 3.2. Since $\text{Hom}_{\mathcal{B}}(\mathbb{C}^n \otimes \mathcal{B}, \mathbb{C}^m \otimes \mathcal{B}) \cong M_{m \times n}(\mathcal{B})$, we can assume that $V \in M_{m \times n}(\mathcal{B})$.

We are now in a position to state the following practical definition of equivariant K_0 -theory.

Definition 3.3. Let \mathcal{U} , \mathcal{B} and σ be as in Definition 3.1. Then $K_0^{\mathcal{U}}(\mathcal{B})$ (resp. ${}^{\mathcal{U}}K_0(\mathcal{B})$) denotes the Grothendieck group obtained from the additive semi-group of Murray-von Neumann equivalent, ad_R° (resp. ad_L°) invariant projections of any size of \mathcal{B} -valued square matrices with addition $[P] + [Q] = [P \oplus Q]$ and additive identity $0 = [0]$ for any size of zero matrix.

Remark 3.4. If $\rho_1^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ and $\rho_2^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^m)$ are finite dimensional $*$ -representations, and $P \in M_{n \times n}(\mathcal{B})$ and $Q \in M_{m \times m}(\mathcal{B})$ are invariant projections, then the notation $P \oplus Q$ refers to the projection $\text{diag}(P, Q)$ in $M_{(n+m) \times (n+m)}(\mathcal{B}) \cong \text{End}(\mathbb{C}^n \oplus \mathbb{C}^m) \otimes \mathcal{B}$, where the representation of \mathcal{U}° on $\mathbb{C}^n \oplus \mathbb{C}^m$ is given by $\rho_1^\circ \oplus \rho_2^\circ$.

Example 3.5. Suppose that \mathcal{B} is a unital $*$ -algebra and σ an automorphism satisfying $\sigma(a)^* = \sigma^{-1}(a^*)$. Then we can define a commutative and co-commutative Hopf $*$ -algebra $\mathcal{U}(\sigma)$ generated by σ with Hopf structure

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \varepsilon(\sigma) = 1, \quad S(\sigma) = \sigma^{-1}$$

and involution $\sigma^* = \sigma$. The left and right actions

$$\sigma \triangleright b = b \triangleleft \sigma = \sigma(b), \quad b \in \mathcal{B},$$

turn \mathcal{B} into a left and right $\mathcal{U}(\sigma)$ -module $*$ -algebra such that, for all $f \in \mathcal{U}$, $\sigma(b \triangleleft f) = \sigma(b) \triangleleft S^{-2}(f)$ and $\sigma(f \triangleright b) = S^2(f) \triangleright \sigma(b)$ since $S^2 = \text{id}$. In this way we obtain a definition of equivariant K_0 -theory which depends only on the automorphism σ . Instead of $K_0^{\mathcal{U}(\sigma)}(\mathcal{B})$ ($= {}^{\mathcal{U}(\sigma)}K_0(\mathcal{B})$), we shall from now on simply write $K_0^\sigma(\mathcal{B})$.

This definition of equivariant K_0 -theory is strongly related to σ -twisted cyclic (co)homology. In particular, as we shall see in Remark 4.5, it allows us to define a pairing between $K_0^\sigma(\mathcal{B})$ and twisted cyclic cohomology.

3.2 Equivariant K_0 -theory and the modular automorphism

In this section we show that, in presence of a modular automorphism, equivariant K_0 -classes are intimately related to unitarily equivalent Hilbert space representations of the opposite crossed product algebra. The computation of the K_0 -group of the standard Podleś sphere can then be reduced to the classification of certain types of unitarily equivalent Hilbert space representations. The details below give also an a posteriori motivation for the definitions made in the previous section.

Throughout this section, we suppose that \mathcal{B} is a left (or right) \mathcal{U} -module $*$ -algebra and $h : \mathcal{B} \rightarrow \mathbb{C}$ is a faithful invariant state with modular automorphism θ . Recall from Section 2.1 that $\langle a, b \rangle := h(a^*b)$ defines an inner product on \mathcal{B} such that $h(a^*(f \triangleright b)) = \langle a, f \triangleright b \rangle = \langle f^* \triangleright a, b \rangle = h((f^* \triangleright a)^*b)$ for all $a, b \in \mathcal{B}$ and $f \in \mathcal{U}$. From

$$h(\theta(a^*)b) = h(ba^*) = \overline{h(ab^*)} = \overline{h(b^*\theta^{-1}(a))} = h(\theta^{-1}(a)^*b),$$

it follows that $\theta(a^*) = \theta^{-1}(a)^*$, and

$$\begin{aligned} h(\theta(f \triangleright b)a^*) &= h(a^*(f \triangleright b)) = h((f^* \triangleright a)^*b) = h(\theta(b)(S(f^*)^* \triangleright a^*)) \\ &= h((S(f^*) \triangleright \theta(b)^*)^*a^*) = h((S^{-2}(f) \triangleright \theta(b))a^*), \end{aligned}$$

implies $\theta(f \triangleright b) = S^{-2}(f) \triangleright \theta(b)$. Similarly one shows that $\theta(b \triangleleft f) = \theta(b) \triangleleft S^2(f)$. Hence $\sigma := \theta^{-1}$ satisfies the conditions of Definition 3.1.

Next we define an inner product on $\mathbb{C}^n \otimes \mathcal{B}$ by

$$\langle v \otimes a, w \otimes b \rangle^\circ := h(ba^*)\langle v, w \rangle_{\mathbb{C}^n}. \quad (28)$$

Note that we used $h(ba^*)$ instead of $h(a^*b)$ so that the right multiplication yields a $*$ -representation of the opposite algebra \mathcal{B}° . If \mathcal{B} is a right \mathcal{U} -module $*$ -algebra and $\rho^\circ : \mathcal{U}^\circ \rightarrow \text{End}(\mathbb{C}^n)$ is a $*$ -representation, then Equation (23) defines a $*$ -representation of the opposite crossed product algebra $(\mathcal{U} \ltimes \mathcal{B})^\circ$.

Similarly, if \mathcal{B} is a left \mathcal{U} -module $*$ -algebra, we set

$$\pi^\circ(f)(v \otimes a) := \rho^\circ(f_{(2)})v \otimes S^{-1}(f_{(1)}) \triangleright a, \quad \pi^\circ(b)(v \otimes a) := v \otimes ab, \quad (29)$$

where $b \in \mathcal{B}$, $f \in \mathcal{U}$ and $v \otimes a \in \mathbb{C}^n \otimes \mathcal{B}$. One easily checks that Equation (29) defines a $*$ -representation of opposite crossed product algebra $(\mathcal{B} \rtimes \mathcal{U})^\circ$, where the inner product on $\mathbb{C}^n \otimes \mathcal{B}$ is given by (28).

By Equation (24), matrix multiplication from the left defines an embedding of $M_{n \times n}(\mathcal{B})$ into $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$. Given $X = (x_{ij})_{i,j=1}^n \in M_{n \times n}(\mathcal{B})$,

let X^+ denote the Hilbert space adjoint of the corresponding operator in $\text{End}(\mathbb{C}^n \otimes \mathcal{B})$. Then, for all $\mathbf{a} = (a_1, \dots, a_n)^t \in \mathcal{B}^n$ and $\mathbf{b} = (b_1, \dots, b_n)^t \in \mathcal{B}^n$,

$$\begin{aligned} \langle \mathbf{a}, X^+ \mathbf{b} \rangle^\circ &= \langle X \mathbf{a}, \mathbf{b} \rangle^\circ = \sum_{i,j=1}^n h(b_j a_i^* x_{ji}^*) = \sum_{i,j=1}^n h(\theta(x_{ji}^*) b_j a_i^*) \\ &= \langle \mathbf{a}, \theta(X^*) \mathbf{b} \rangle^\circ. \end{aligned}$$

Thus $X^+ = \theta(X^*) = \sigma(X)^* = X^\dagger$ and the above embedding becomes a $*$ -representation. In particular, projections in $M_{n \times n}(\mathcal{B})$ yield orthogonal projections on $\mathbb{C}^n \otimes \mathcal{B}$.

The following proposition relates invariant projections to Hilbert space representations of the opposite crossed product algebra.

Proposition 3.6. *Let $P \in M_{n \times n}(\mathcal{B})$ be a projection. Then the restriction of π° to the projective right \mathcal{B} -module $P\mathcal{B}^n$ defines a $*$ -representation of $(\mathcal{U} \ltimes \mathcal{B})^\circ$ (or $(\mathcal{B} \rtimes \mathcal{U})^\circ$) if and only if P is invariant.*

Proof. We prove Proposition 3.6 for left \mathcal{U} -module $*$ -algebras, the proof for the right-handed counterpart is similar.

For column vectors $\mathbf{b} = (b_1, \dots, b_n)^t \in \mathcal{B}^n \cong \mathbb{C}^n \otimes \mathcal{B}$, the first equation of (29) can be written $\pi^\circ(f)\mathbf{b} = \rho^\circ(f_{(2)})(S^{-1}(f_{(1)}) \triangleright \mathbf{b})$. Thus

$$\begin{aligned} \pi^\circ(f)(P\mathbf{b}) &= \rho^\circ(f_{(3)})(S^{-1}(f_{(2)}) \triangleright P)(S^{-1}(f_{(1)}) \triangleright \mathbf{b}) \\ &= \rho^\circ(f_{(5)})(S^{-1}(f_{(4)}) \triangleright P)\rho^\circ(S(f_{(3)}))\rho^\circ(f_{(2)})(S^{-1}(f_{(1)}) \triangleright \mathbf{b}) \\ &= \text{ad}_R^\circ(S(f_{(2)}))(P)\pi^\circ(f_{(1)})\mathbf{b}. \end{aligned}$$

Hence, as Hilbert space operators on \mathcal{B}^n ,

$$\pi^\circ(f)P = \text{ad}_R^\circ(S(f_{(2)}))(P)\pi^\circ(f_{(1)}).$$

Since $\text{ad}_R^\circ(S(f))(P) = \text{ad}_L^\circ(f)(P) = \pi^\circ(f_{(1)})P\pi^\circ(S^{-1}(f_{(2)}))$, it follows that $\pi^\circ(f)P = P\pi^\circ(f)$ if and only if $\text{ad}_R^\circ(f)(P) = \varepsilon(f)P$. \square

3.3 The equivariant K_0 -group of the standard Podleś sphere

We restrict ourselves to the computation of $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2))$ with respect to the inverse modular automorphism θ^{-1} of $\mathcal{O}(S_q^2)$, the computation of its right-handed counterpart is analogous. In particular, the outcome would be that $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2)) \cong \mathcal{U}_q(\text{su}_2)K_0(\mathcal{O}(S_q^2))$.

Our first aim is to construct representatives for equivariant K_0 -classes. For $n \in \frac{1}{2}\mathbb{Z}$ and $l = |n|, |n| + 1, \dots$, let \mathbf{t}_n^l denote the row vector

$$\mathbf{t}_n^l := (t_{n,-l}^l, t_{n,-l+1}^l, \dots, t_{n,l}^l), \quad (30)$$

where $t_{n,k}^l$ are the matrix elements from Section 2.2. Recall from Section 2.2 that there is a $(2l+1)$ -dimensional $*$ -representation of $\mathcal{U}_q(\mathfrak{su}_2)$ on

$$V_n^l := \text{span}\{t_{n,k}^l : k = -l, \dots, l\} = \text{span}\{v_{n,k}^l : k = -l, \dots, l\}, \quad l \geq |n|,$$

given by Equation (10). These representations are irreducible and called spin- l -representations. Let $\sigma_l : \mathcal{U}_q(\mathfrak{su}_2) \rightarrow M_{(2l+1) \times (2l+1)}(\mathbb{C})$ be the matrix representation determined by $f \triangleright \sum_{k=-l}^l \alpha_k v_{nk}^l = \sum_{k,j=-l}^l \sigma_l(f)_{jk} \alpha_k v_{nj}^l$. Then

$$f \triangleright \mathbf{t}_n^l = \mathbf{t}_n^l \sigma_l(f), \quad f \triangleright \mathbf{t}_n^{l*} = \sigma_l(S(f)) \mathbf{t}_n^{l*}, \quad f \in \mathcal{U}_q(\mathfrak{su}_2), \quad (31)$$

where we used (1) in the second relation. Consider the homomorphism

$$\rho_l^\circ : \mathcal{U}_q(\mathfrak{su}_2)^\circ \rightarrow \text{End}(\mathbb{C}^{2l+1}), \quad \rho_l^\circ(f) := \sigma_l(K^{-1}S(f)K). \quad (32)$$

From $K^2 f K^{-2} = S^2(f)$ and $S(f)^* = S^{-1}(f^*)$ for all $f \in \mathcal{U}_q(\mathfrak{su}_2)$, it follows that ρ_l° is a $*$ -representation of $\mathcal{U}_q(\mathfrak{su}_2)^\circ$.

Finally, for $N \in \mathbb{Z}$, define

$$P_N := \rho_{|N|/2}^\circ(K^{-1}) \mathbf{t}_{N/2}^{|N|/2*} \mathbf{t}_{N/2}^{|N|/2} \rho_{|N|/2}^\circ(K). \quad (33)$$

We summarize some crucial properties of these matrices in the next lemma.

Lemma 3.7. *The matrices P_N belong to $M_{|N|+1 \times |N|+1}(\mathcal{O}(S_q^2))$ and are ad_R° -invariant projections with respect to the anti-representation $\rho_{|N|/2}^\circ$ of $\mathcal{U}_q(\mathfrak{su}_2)$ and the involution $P_N^\dagger = \theta^{-1}(P_N)^*$, where $\theta^{-1}(b) = K^2 \triangleright b$ for all $b \in \mathcal{O}(S_q^2)$.*

Proof. For brevity of notation, set $n := N/2$ and $l := |N|/2$. From

$$\partial_{K^2}(t_{n,j}^{l*} t_{n,k}^l) = (\partial_{K^{-2}}(t_{n,j}^l))^* \partial_{K^2}(t_{n,k}^l) = t_{n,j}^{l*} t_{n,k}^l,$$

it follows that the entries of P_N belong to $\mathcal{O}(S_q^2)$. Using $K^2 f K^{-2} = S^2(f)$ and Equations (27) and (31)–(33), we get

$$\begin{aligned} \text{ad}_R^\circ(f)(P_N) &= \rho_l^\circ(S^{-1}(f_{(1)}))(S^{-2}(f_{(2)}) \triangleright P_N) \rho_l^\circ(f_{(3)}) \\ &= \sigma_l(K^{-1} f_{(1)} K^2)(S^{-2}(f_{(2)}) \triangleright \mathbf{t}_n^{l*})(S^{-2}(f_{(3)}) \triangleright \mathbf{t}_n^l) \sigma_l(K^{-2} S(f_{(4)}) K) \\ &= \sigma_l(K^{-1} f_{(1)} K^2 S^{-1}(f_{(2)})) \mathbf{t}_n^{l*} \mathbf{t}_n^l \sigma_l(S^{-2}(f_{(3)}) K^{-2} S(f_{(4)}) K) \\ &= \sigma_l(K^{-1} f_{(1)} S(f_{(2)}) K^2) \mathbf{t}_n^{l*} \mathbf{t}_n^l \sigma_l(K^{-2} f_{(3)} S(f_{(4)}) K) \\ &= \varepsilon(f) \sigma_l(K) \mathbf{t}_n^{l*} \mathbf{t}_n^l \sigma_l(K^{-1}) = \varepsilon(f) P_N \end{aligned}$$

for all $f \in \mathcal{U}_q(\mathfrak{su}_2)$. Thus P_N is ad_R° -invariant. The ad_R° -invariance implies

$$\theta^{-1}(P_N) = K^2 \triangleright P_N = \rho_l^\circ(K^2) \text{ad}_R^\circ(K^2)(P_N) \rho_l^\circ(K^{-2}) = \rho_l^\circ(K^2) P_N \rho_l^\circ(K^{-2}),$$

hence $P_N^\dagger := \theta^{-1}(P_N)^* = (\rho_l^\circ(K) \mathbf{t}_n^{l*} \mathbf{t}_n^l \rho_l^\circ(K^{-1}))^* = P_N$. Clearly, $P_N^2 = P_N$ since $\mathbf{t}_n^l \mathbf{t}_n^{l*} = 1$ by (7), so P_N is a projection. \square

By Definition 3.3, the projections in the last lemma represent classes of $K_0^{\mathcal{U}_q(\mathfrak{su}_2)}(\mathcal{O}(S_q^2))$ relative to the inverse modular automorphism θ^{-1} of $\mathcal{O}(S_q^2)$. The next proposition shows that the projections P_N determine completely $K_0^{\mathcal{U}_q(\mathfrak{su}_2)}(\mathcal{O}(S_q^2))$.

Proposition 3.8. *With respect to the automorphism $\theta^{-1} : \mathcal{B} \rightarrow \mathcal{B}$ given by $\theta^{-1}(b) = K^2 \triangleright b$, the group $K_0^{\mathcal{U}_q(\mathfrak{su}_2)}(\mathcal{O}(S_q^2))$ is isomorphic to the free abelian group with one generator, $[P_N]$, for each $N \in \mathbb{Z}$.*

Proof. By Proposition 3.6, the projections P_N determine Hilbert space representations of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$. Our first aim is to show that, for $N \in \mathbb{Z}$, we obtain pairwise inequivalent integrable representations and that each Hilbert space representations arising from ad_R° -invariant $\mathcal{O}(S_q^2)$ -valued projections decomposes into those obtained by P_N . Here, a $*$ -representation of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ is called *integrable* if its restriction to $\mathcal{U}_q(\mathfrak{su}_2)^\circ$ decomposes into finite dimensional $*$ -representations. Note that, by the Clebsch-Gordon decomposition, the (tensor product) representation of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ on $\mathbb{C}^n \otimes \mathcal{O}(S_q^2)$ defined in (29) is integrable and so are the representations from Proposition 3.6.

Straightforward calculations show that $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ is isomorphic to $\mathcal{O}(S_{q^{-1}}^2) \rtimes \mathcal{U}_{q^{-1}}(\mathfrak{su}_2)$ with A replaced by $q^{-2}A$. By using alternatively the opposite algebras and the replacement $q \mapsto q^{-1}$, we can apply freely the results from [SW3].

To begin, consider the Hopf fibration of $\mathcal{O}(\text{SU}_q(2))^\circ$ given by

$$\mathcal{O}(\text{SU}_q(2))^\circ = \oplus_{N \in \mathbb{Z}} M_N^\circ, \quad M_N^\circ := \{x \in \mathcal{O}(\text{SU}_q(2))^\circ : \partial_{K^2}(x) = q^{-N}x\}.$$

As in [SW3], the restriction of the GNS-representation π_h to M_N° defines pairwise inequivalent integrable representations of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$, and each irreducible integrable $*$ -representation of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ is unitarily equivalent to one on M_N° . Moreover, the integrable $*$ -representations of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ on M_N° and on $P_N \mathcal{O}(S_q^2)^{|N|+1} = (((\mathcal{O}(S_q^2)^\circ)^{|N|+1})^t \circ (P_N)^t)^t$ are unitarily equivalent, where \circ stands for the opposite multiplication.

With $n \in \mathbb{N}$, let $P \in M_{n \times n}(\mathcal{O}(S_q^2))$ be an invariant projection. From [SW3, Theorem 4.1] and the preceding, we conclude that the integrable representation of $(\mathcal{O}(S_q^2) \rtimes \mathcal{U}_q(\mathfrak{su}_2))^\circ$ on $P \mathcal{O}(S_q^2)^n$ is equivalent to the direct sum of irreducible representations on $(\oplus_{N=-N_0}^{N_0} \oplus_{i=1}^{n_N(P)} P_N) \mathcal{O}(S_q^2)^m$, where $N_0 \in \mathbb{N}$ is sufficiently large and the orthogonal sum of projections is given as in Remark 3.4. The numbers $n_N(P) \in \mathbb{N}_0$ denote multiplicities and $m = \sum_{N=-N_0}^{N_0} n_N(P)(|N| + 1)$. Set $Q := \oplus_{N=-N_0}^{N_0} \oplus_{i=1}^{n_N(P)} P_N$ and let $U : P \mathcal{O}(S_q^2)^n \rightarrow Q \mathcal{O}(S_q^2)^m$ denote the unitary operator realizing the equivalence.

Then the operator $QUP \in \text{Hom}_{\mathcal{O}(\mathbb{S}_q^2)}(\mathbb{C}^n \otimes \mathcal{O}(\mathbb{S}_q^2), \mathbb{C}^m \otimes \mathcal{O}(\mathbb{S}_q^2))$ establishes a Murray-von Neumann equivalence between P and Q .

Now let P and P' be invariant $\mathcal{O}(\mathbb{S}_q^2)$ -valued projections. The proof of [SW3, Theorem 4.1] shows that P and P' are Murray-von Neumann equivalent if and only if $n_N(P) = n_N(P')$ for all $N \in \mathbb{Z}$ in their decompositions described in the last paragraph. From this, it follows first that $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(\mathbb{S}_q^2))$ has cancellation, and second that $[P] - [P'] = 0$ if and only if $n_N(P) = n_N(P')$ for all $N \in \mathbb{Z}$. Hence $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(\mathbb{S}_q^2))$ is isomorphic to the free abelian group generated by $\{[P_N] : N \in \mathbb{Z}\}$. \square

Later, the index pairing in Proposition 5.1 confirms that (for transcendental q) there are no relations between the generators $[P_N]$ of the equivariant K_0 -group $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(\mathbb{S}_q^2))$.

4 Twisted Chern character

The twisted Chern character will be defined as a map from equivariant K_0 -theory to twisted cyclic homology. To do so, we need a convenient description of twisted cyclic homology.

For a complex unital algebra \mathcal{A} with automorphism $\lambda : \mathcal{A} \rightarrow \mathcal{A}$, set $C_n := \mathcal{A}^{\otimes(n+1)}$ and define

$$d_{n,i}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \quad i \neq n, \quad (34)$$

$$d_{n,n}^\lambda(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \lambda(a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \quad (35)$$

$$\tau_n^\lambda(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \lambda(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}, \quad (36)$$

$$s_{n,i}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n, \quad (37)$$

where $n \in \mathbb{N}_0$ and $0 \leq i \leq n$. For $\lambda = \text{id}$, these are the face, cyclic and degeneracy operators of the standard cyclic object associated to \mathcal{A} [Lod]. For general λ , the operator $(\tau_n^\lambda)^{n+1}$ acting on C_n by

$$(\tau_n^\lambda)^{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \lambda(a_0) \otimes \lambda(a_1) \otimes \cdots \otimes \lambda(a_n)$$

fails to be the identity. To obtain a cyclic object, one passes to the cokernels $C_n^\lambda := C_n / \text{im}(\text{id} - (\tau_n^\lambda)^{n+1})$. The twisted cyclic homology $\text{HC}_*^\lambda(\mathcal{A})$ is now the

total homology of Connes' mixed (b, B) -bicomplex $\text{BC}^\lambda(\mathcal{A})$:

$$\begin{array}{ccccccc}
b_4^\lambda \downarrow & & b_3^\lambda \downarrow & & b_2^\lambda \downarrow & & b_1^\lambda \downarrow \\
C_3^\lambda & \xleftarrow{B_2^\lambda} & C_2^\lambda & \xleftarrow{B_1^\lambda} & C_1^\lambda & \xleftarrow{B_0^\lambda} & C_0^\lambda \\
b_3^\lambda \downarrow & & b_2^\lambda \downarrow & & b_1^\lambda \downarrow & & \\
C_2^\lambda & \xleftarrow{B_1^\lambda} & C_1^\lambda & \xleftarrow{B_0^\lambda} & C_0^\lambda & & \\
b_2^\lambda \downarrow & & b_1^\lambda \downarrow & & & & \\
C_1^\lambda & \xleftarrow{B_0^\lambda} & C_0^\lambda & & & & \\
b_1^\lambda \downarrow & & & & & & \\
C_0^\lambda & & & & & &
\end{array} \tag{38}$$

The boundary maps b_n^λ and B_n^λ are given by

$$b_n^\lambda = \sum_{i=0}^{n-1} (-1)^i d_{n,i} + (-1)^n d_{n,n}^\lambda, \quad B_n^\lambda = (1 - (-1)^{n+1} \tau_{n+1}) s_n N_n^\lambda, \tag{39}$$

where $N_n^\lambda = \sum_{j=0}^n (-1)^{nj} (\tau_n^\lambda)^j$, and the maps

$$s_n : C_n^\lambda \rightarrow C_{n+1}^\lambda, \quad s_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \tag{40}$$

are called “extra degeneracies”.

The homology of the columns is the twisted Hochschild homology of \mathcal{A} and denoted by $\text{HH}_*^\lambda(\mathcal{A})$. An element $\eta \in \mathcal{A}^{\otimes(n+1)}$ such that $b_n^\lambda(\eta) = 0$ is called a *twisted Hochschild n -cycle*. The class of η in C_n^λ defines an element in twisted Hochschild and cyclic homology by putting it at the $(0, n)$ -th position of the (b, B) -bicomplex (38).

By dualizing, one passes from twisted cyclic homology $\text{HC}_*^\lambda(\mathcal{A})$ to twisted cyclic cohomology $\text{HC}_\lambda^*(\mathcal{A})$. A complex for computing $\text{HC}_\lambda^*(\mathcal{A})$ is obtained by applying the functor $\text{Hom}(\cdot, \mathbb{C})$ to each entry of the (b, B) -complex.

An alternative description of twisted cyclic cohomology $\text{HC}_\lambda^*(\mathcal{A})$ is as follows [KMT]. Let $C_\lambda^n(\mathcal{A})$ denote the space of $(n+1)$ -linear forms ϕ on \mathcal{A} such that $\phi = (-1)^n \tau_n^{\lambda*} \phi$, where

$$\tau_n^{\lambda*} \phi(a_0, \dots, a_n) = \phi(\lambda(a_n), a_0, \dots, a_{n-1}).$$

With the coboundary operator $b_n^{\lambda*} : C_\lambda^{n-1}(\mathcal{A}) \rightarrow C_\lambda^n(\mathcal{A})$,

$$\begin{aligned}
b_n^{\lambda*} \phi(a_0, \dots, a_n) &= \sum_{j=0}^{n-1} (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_n) \\
&\quad + (-1)^n \phi(\lambda(a_n) a_0, \dots, a_{n-1}),
\end{aligned}$$

one gets a cochain complex whose homology is isomorphic to $\mathrm{HC}_\lambda^*(\mathcal{A})$. The elements $\phi \in C_\lambda^n(\mathcal{A})$ satisfying $b_{n+1}^{\lambda*}\phi = 0$ are called *twisted cyclic n -cocycles*. An isomorphism between the two versions of twisted cyclic cohomology is given by putting a twisted cyclic n -cocycle at the $(0, n)$ -th position in the dual (b, B) -complex and zeros elsewhere.

Evaluating cycles on cocycles yields a dual pairing between $\mathrm{HC}_\lambda^*(\mathcal{A})$ and $\mathrm{HC}_\lambda^*(\mathcal{A})$. For $\lambda = \mathrm{id}$, there is a Chern character map from K -theory to cyclic homology available. Composing the Chern character with the evaluation on cocycles defines a pairing between K -theory and cyclic cohomology (the Chern-Connes pairing). Our aim is to construct a similar pairing between equivariant K_0 -theory and twisted cyclic cohomology. The primary tool will be a “twisted” Chern character from the equivariant K_0 -group to even twisted cyclic homology. This shall be our concern in the remainder of this section.

The discussion will be restricted to the following setting: We assume that \mathcal{U} is a Hopf $*$ -algebra, \mathcal{B} a unital left \mathcal{U} -module $*$ -algebra, and the automorphism $\lambda : \mathcal{B} \rightarrow \mathcal{B}$ can be described by a group-like element $k \in \mathcal{U}$, i.e., $\Delta(k) = k \otimes k$ and $\lambda(b) = k \triangleright b$ for all $b \in \mathcal{B}$. Note that $\varepsilon(k) = 1$ and $S(k) = k^{-1}$. In particular, it follows that $\lambda(b)^* = \lambda^{-1}(b^*)$.

The link between the “non-twisted” and the “twisted” case is provided by the so-called quantum trace. Given matrices $A_k = (a_{i_k, j_k}^k) \in M_{m \times m}(\mathcal{B})$, and an (anti-)representation $\rho^\circ : \mathcal{U} \rightarrow \mathrm{End}(\mathbb{C}^m)$, we define the quantum trace Tr_λ by

$$\mathrm{Tr}_\lambda(A_0 \otimes A_1 \otimes \cdots \otimes A_n) = \sum_{j_0, \dots, j_{n+1}} \rho^\circ(k)_{j_{n+1}, j_0} a_{j_0, j_1}^0 \otimes a_{j_1, j_2}^1 \otimes \cdots \otimes a_{j_n, j_{n+1}}^n. \quad (41)$$

Lemma 4.1. *With the conventions introduced above, let \mathcal{A} denote the algebra of ad_R° -invariant matrices belonging to $M_{m \times m}(\mathcal{B})$. Then Tr_λ defines a morphism of complexes from $\mathrm{BC}^{\mathrm{id}}(\mathcal{A})$ to $\mathrm{BC}^\lambda(\mathcal{B})$.*

Proof. To prove the lemma, it suffices to show that Tr_λ intertwines the operators defined in Equations (34)–(37) and (40). For the face operators $d_{n,i}$, $i \neq n$, and the degeneracy operators $s_{n,i}$ and s_n , the assertion is obvious. For $d_{n,n}^\lambda$ and τ_n^λ , one uses the fact that

$$\rho^\circ(k)A = \lambda(A)\rho^\circ(k) \quad \text{for all } A \in \mathcal{A},$$

since

$$A = \mathrm{ad}_R^\circ(k)(A) = \rho^\circ(k^{-1})(k \triangleright A)\rho^\circ(k) = \rho^\circ(k^{-1})\lambda(A)\rho^\circ(k)$$

for any ad_R° -invariant matrix $A \in M_{m \times m}(\mathcal{B})$. □

The next proposition introduces a twisted version of the Chern character mapping equivariant K_0 -groups into twisted cyclic homology.

Proposition 4.2. *Let \mathcal{U} , \mathcal{B} , λ , ρ° be as above. Suppose that there is an automorphism of \mathcal{B} satisfying the conditions of Definition 3.1. For any invariant projection $P \in M_{m \times m}(\mathcal{B})$, set*

$$\text{ch}_{2n}^\lambda(P) := (-1)^n \frac{(2n)!}{n!} \text{Tr}_\lambda(P \otimes P \otimes \cdots \otimes P) \in \mathcal{B}^{\otimes 2n+1}.$$

Then there are well-defined additive maps $\text{ch}_{0,n}^\lambda : K_0^\mathcal{U}(\mathcal{B}) \rightarrow \text{HC}_{2n}^\lambda(\mathcal{B})$ given by

$$\text{ch}_{0,n}^\lambda([P]) = (\text{ch}_{2n}^\lambda(P), \dots, \text{ch}_0^\lambda(P)).$$

Proof. Let $P \in M_{m \times m}(\mathcal{B})$ be an invariant projection. From the Chern character of K_0 -theory with values in (non-twisted) cyclic homology (cf. [Lod]), it is known that $(\text{ch}_{2n}^{\text{id}}(P), \dots, \text{ch}_0^{\text{id}}(P))$ defines a cycle in $\text{BC}^{\text{id}}(\mathcal{A})$, where \mathcal{A} denotes again the subalgebra of ad_R° -invariant matrices in $M_{m \times m}(\mathcal{B})$. By Lemma 4.1, $(\text{ch}_{2n}^\lambda(P), \dots, \text{ch}_0^\lambda(P))$ is a cycle in $\text{BC}^\lambda(\mathcal{B})$. Moreover, the additivity of $\text{ch}_{0,n}^\lambda$ follows from the additivity of Tr_λ .

It remains to prove that the homology class of $\text{ch}_{0,n}^\lambda([P])$ does not depend on representatives. Let $P \in M_{j \times j}(\mathcal{B})$ and $Q \in M_{k \times k}(\mathcal{B})$ be invariant projections with respect to the anti-representations $\rho_1^\circ : \mathcal{U} \rightarrow \text{End}(\mathbb{C}^j)$ and $\rho_2^\circ : \mathcal{U} \rightarrow \text{End}(\mathbb{C}^k)$, respectively, and suppose that P and Q are Murray-von Neumann equivalent. By considering the direct sum $\rho_1^\circ \oplus \rho_2^\circ$ on \mathbb{C}^{j+k} , we may assume that P and Q belong to the same matrix algebra $M_{m \times m}(\mathcal{B})$ and that there is an invertible element $U \in M_{m \times m}(\mathcal{B})$ such that $UPU^{-1} = Q$. To be more precise, we consider

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \sim P, \quad \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \sim Q, \quad U := \begin{pmatrix} 1 - P & PV^\dagger Q \\ QVP & 1 - Q \end{pmatrix} = U^{-1},$$

where $V \in \text{Hom}_{\mathcal{B}}(\mathbb{C}^j \otimes \mathcal{B}, \mathbb{C}^k \otimes \mathcal{B})$ establishes the Murray-von Neumann equivalence between P and Q . Clearly, P and Q are ad_R° -invariant with respect to $\rho_1^\circ \oplus \rho_2^\circ$ and, by Definition 3.1, so is U . Since conjugation acts as the identity on cyclic homology [Lod, Proposition 4.1.2], $(\text{ch}_{2n}^{\text{id}}(P), \dots, \text{ch}_0^{\text{id}}(P))$ and $(\text{ch}_{2n}^{\text{id}}(Q), \dots, \text{ch}_0^{\text{id}}(Q))$ differ in $\text{BC}^{\text{id}}(\mathcal{A})$ only by a boundary. Hence, by Lemma 4.1, the Chern characters $\text{ch}_{0,n}^\lambda([P])$ in $\text{HC}_{2n}^\lambda(\mathcal{B})$ do not depend on the representative of the class $[P]$ in $K_0^\mathcal{U}(\mathcal{B})$. \square

Corollary 4.3. *There is a pairing*

$$\langle \cdot, \cdot \rangle : \text{HC}_{2n}^\lambda(\mathcal{B}) \times K_0^\mathcal{U}(\mathcal{B}) \rightarrow \mathbb{C}$$

given by evaluating cocycles on $\text{ch}_{0,n}^\lambda$. For a twisted cyclic $2n$ -cocycle Φ , it takes the form

$$\langle [\Phi], [P] \rangle = (-1)^n \frac{(2n)!}{n!} \Phi(\text{Tr}_\lambda(P \otimes P \otimes \cdots \otimes P)). \quad (42)$$

Proof. The first assertion follows from Proposition 4.2, the second from the embedding of cycles $\Phi \in C_\lambda^n(\mathcal{A})$ satisfying $b_{n+1}^{\lambda*} \Phi = 0$ into the (b, B) -complex. \square

Remark 4.4. Note that the pairing between Φ and Chern character ch_{2n}^λ is compatible with Connes' periodicity operator $\mathcal{S} : \text{HC}_\lambda^n(\mathcal{B}) \rightarrow \text{HC}_\lambda^{n+2}(\mathcal{B})$, that is,

$$\mathcal{S}\Phi(\text{ch}_{2n+2}^\lambda(P)) = \Phi(\text{ch}_{2n}^\lambda(P)) \quad (43)$$

for any invariant idempotent P , where

$$\begin{aligned} \mathcal{S}\Phi(a_0, \dots, a_{n+2}) := & -\frac{1}{(n+1)(n+2)} \left(\sum_{i=1}^{n+1} \phi(a_0, \dots, a_{i-1}a_i a_{i+1}, \dots, a_{n+2}) \right. \\ & \left. - \sum_{1 \leq i < j \leq (n+1)} (-1)^{i+j} \phi(a_0, \dots, a_{i-1}a_i, \dots, a_j a_{j+1}, \dots, a_{n+2}) \right). \end{aligned}$$

Actually, Corollary 4.3 defines a pairing between periodic twisted cyclic cohomology and equivariant K_0 -theory, but we shall not go into the details.

Remark 4.5. The description of $\text{HC}_\lambda^*(\mathcal{B})$ involves only an algebra \mathcal{B} with an automorphism λ . If \mathcal{B} is a $*$ -algebra and λ satisfies $\lambda(a)^* = \lambda^{-1}(a^*)$, then, by Example 3.5, we have a definition of equivariant K_0 -theory $K_0^\lambda(\mathcal{B})$ and, by Corollary 4.3, a pairing between $\text{HC}_\lambda^*(\mathcal{B})$ and $K_0^\lambda(\mathcal{B})$.

5 Chern-Connes pairing

5.1 The modular automorphism case

This section is devoted to the calculation of the pairing

$$\langle \cdot, \cdot \rangle : \text{HC}_\theta^{2n}(\mathcal{O}(S_q^2)) \times K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2)) \rightarrow \mathbb{C} \quad (44)$$

described in Corollary 4.3, where θ denotes the modular automorphism from Equation (19). This includes in particular the pairing of $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2))$ with the θ -twisted cyclic 2-cocycle τ associated with the volume form of the distinguished 2-dimensional covariant differential calculus on $\mathcal{O}(S_q^2)$ (cf. Proposition 2.3). Note, by the way, that $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2))$ is given in Proposition 3.8 with respect to the inverse automorphism θ^{-1} .

The twisted cyclic homology of $\mathcal{O}(S_q^2)$ was computed by Hadfield [Had]. Dualizing, we conclude from [Had] that

$$\mathrm{HC}_\theta^{2n}(\mathcal{O}(S_q^2)) = \mathbb{C}[\mathcal{S}^n \varepsilon] \oplus \mathbb{C}[\mathcal{S}^n h], \quad \mathrm{HC}_\theta^{2n+1}(\mathcal{O}(S_q^2)) = 0, \quad n \in \mathbb{N}_0,$$

where h is the Haar state on $\mathcal{O}(S_q^2)$, ε is the restriction of the counit of $\mathcal{O}(\mathrm{SU}_q(2))$ to $\mathcal{O}(S_q^2)$, and $\mathcal{S} : \mathrm{HC}_\theta^n(\mathcal{O}(S_q^2)) \rightarrow \mathrm{HC}_\theta^{n+2}(\mathcal{O}(S_q^2))$ denotes Connes' periodicity operator from Remark 4.4. As a consequence, the pairing in (44) is completely determined by the pairing of $\mathrm{HC}_\theta^0(\mathcal{O}(S_q^2))$ with $K_0^{\mathcal{U}_q(\mathrm{su}_2)}(\mathcal{O}(S_q^2))$.

Proposition 5.1. *The pairing between $\mathrm{HC}_\theta^0(\mathcal{O}(S_q^2))$ and $K_0^{\mathcal{U}_q(\mathrm{su}_2)}(\mathcal{O}(S_q^2))$ is given by*

$$\langle [\varepsilon], [P_N] \rangle = q^N, \quad \langle [h], [P_N] \rangle = q^{-N}, \quad N \in \mathbb{Z}.$$

Proof. Set $n := N/2$ and $l := |N|/2$. With the notation of Section 3.3, it follows from (31) and (32) that $\mathbf{t}_n^l \rho_l^\circ(K^{-2}) = K^2 \triangleright \mathbf{t}_n^l$. Applying successively (42), (33), (41), (10) and (7), we get

$$\begin{aligned} \langle [\varepsilon], [P_N] \rangle &= \varepsilon(\mathrm{ch}_0^\theta(P_N)) = \varepsilon(\mathrm{Tr} \rho_l^\circ(K^{-3}) \mathbf{t}_n^{l*} \mathbf{t}_n^l \rho_l^\circ(K)) = \varepsilon(\mathrm{Tr} \mathbf{t}_n^{l*} (K^2 \triangleright \mathbf{t}_n^l)) \\ &= \sum_{j=-l}^l q^{2j} \varepsilon(t_{n,j}^{l*} t_{n,j}^l) = q^{2n}. \end{aligned}$$

Note that $\theta(t_{n,j}^l) = q^{-2j-2n} t_{n,j}^l$ by (18) and $\sum_{j=-l}^l t_{n,j}^l t_{n,j}^{l*} = 1$ by (7). Analogously to the above, we have

$$\langle [h], [P_N] \rangle = \sum_{j=-l}^l q^{2j} h(t_{n,j}^{l*} t_{n,j}^l) = q^{-2n} \sum_{j=-l}^l h(t_{n,j}^l t_{n,j}^{l*}) = q^{-2n}. \quad \square$$

Now we compute the pairing of $K_0^{\mathcal{U}_q(\mathrm{su}_2)}(\mathcal{O}(S_q^2))$ with the θ -twisted cyclic 2-cycle τ from Proposition 2.3.

Proposition 5.2. *The pairing of $[\tau]$ with $K_0^{\mathcal{U}_q(\mathrm{su}_2)}(\mathcal{O}(S_q^2))$ is given by*

$$\langle [\tau], [P_N] \rangle = 2[N]_q, \quad N \in \mathbb{Z}. \quad (45)$$

In particular, $\tau = \frac{2}{q^{-1}-q}(\mathcal{S}h - \mathcal{S}\varepsilon)$.

Proof. Let $n = N/2 > 0$. Inserting (22) and (33) into (42), we get

$$\langle [\tau], [P_N] \rangle = 2h(\mathrm{Tr}_\theta \mathbf{t}_n^{n*} \mathbf{t}_n^n (q^{-1} \partial_F(\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_E(\mathbf{t}_n^{n*} \mathbf{t}_n^n) - q \partial_E(\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_F(\mathbf{t}_n^{n*} \mathbf{t}_n^n))).$$

From (9), (11) and (12), it follows that

$$\partial_E(\mathbf{t}_n^{n*} \mathbf{t}_n^n) = -q^{n-1} \alpha_{n-1}^n \mathbf{t}_n^{n*} \mathbf{t}_{n-1}^n, \quad \partial_F(\mathbf{t}_n^{n*} \mathbf{t}_n^n) = q^n \alpha_{n-1}^n \mathbf{t}_{n-1}^{n*} \mathbf{t}_n^n. \quad (46)$$

Equation (8) implies $\mathbf{t}_n^n \mathbf{t}_n^{n*} = \mathbf{t}_{n-1}^n \mathbf{t}_{n-1}^{n*} = 1$ and $\mathbf{t}_n^n \mathbf{t}_{n-1}^{n*} = 0$. Hence

$$\langle [\tau], [P_N] \rangle = 2q^{2n} (\alpha_{n-1}^n)^2 h(\mathrm{Tr}_\theta \mathbf{t}_n^{n*} \mathbf{t}_n^n) = 2q^{2n} (\alpha_{n-1}^n)^2 h(\mathrm{Tr}_\theta P_N).$$

Applying $h(\mathrm{Tr}_\theta P_N) = \langle [h], [P_N] \rangle = q^{-2n}$ and $(\alpha_{n-1}^n)^2 = [2n]_q$ proves (45) for $N > 0$. If $n = N/2 < 0$, Equation (46) becomes

$$\partial_E(\mathbf{t}_n^{|n|*} \mathbf{t}_n^{|n|}) = q^n \alpha_n^{|n|} \mathbf{t}_{n+1}^{|n|*} \mathbf{t}_n^{|n|}, \quad \partial_F(\mathbf{t}_n^{|n|*} \mathbf{t}_n^{|n|}) = -q^{n+1} \alpha_n^{|n|} \mathbf{t}_n^{|n|*} \mathbf{t}_{n+1}^{|n|}, \quad (46')$$

and by the same arguments as above, we get

$$\langle [\tau], [P_N] \rangle = -2q^{2n} (\alpha_n^{|n|})^2 \langle [h], [P_N] \rangle = -2[2|n|]_q = 2[N]_q.$$

The case $N = 0$ is trivial since $\partial_F(1) = \partial_E(1) = 0$.

It has been shown in [Had] that $\tau = \beta(\mathcal{S}h - \mathcal{S}\varepsilon)$ with $\beta \in \mathbb{C} \setminus \{0\}$. By the preceding, Equation (43) and Proposition 5.1,

$$\begin{aligned} 2[N]_q &= \tau(\mathrm{ch}_2^\theta(P_N)) = \beta \mathcal{S}(h - \varepsilon)(\mathrm{ch}_2^\theta(P_N^N)) = \beta(h - \varepsilon)(\mathrm{ch}_0^\theta(P_N^N)) \\ &= \beta(q^{-N} - q^N), \end{aligned}$$

thus $\beta = \frac{2}{q^{-1}-q}$. □

5.2 The case of the inverse modular automorphism

The modular automorphism θ does not correspond to the “no dimension drop” case of twisted Hochschild homology since $\mathrm{HH}_2^\theta(\mathcal{O}(\mathbb{S}_q^2)) = 0$ (see [Had]). On the other hand, as shown in [Had], the dimension drop can be avoided by taking the inverse modular automorphism. In this case, $\mathrm{HH}_2^{\theta^{-1}}(\mathcal{O}(\mathbb{S}_q^2)) \cong \mathbb{C}$ and $\mathrm{HC}_2^{\theta^{-1}}(\mathcal{O}(\mathbb{S}_q^2)) \cong \mathbb{C}^2$. By duality, we have $\mathrm{HC}_{\theta^{-1}}^2(\mathcal{O}(\mathbb{S}_q^2)) \cong \mathbb{C}^2$.

Observe that any λ -cyclic n -cocycle $\psi \in \mathrm{HC}_\lambda^n(\mathcal{A})$ defines a functional on $\mathrm{HH}_n^\lambda(\mathcal{A})$ since $\psi \circ b_{n+1}^\lambda = b_{n+1}^{\lambda*} \psi = 0$ (cf. Section 4). One generator of $\mathrm{HC}_{\theta^{-1}}^2(\mathcal{O}(\mathbb{S}_q^2))$ is given by $[\mathcal{S}\varepsilon]$. It is easily shown by replacing Tr_θ by $\mathrm{Tr}_{\theta^{-1}}$ in the previous section that $\langle [\mathcal{S}\varepsilon], [P_N] \rangle = \varepsilon(\mathrm{ch}_0^{\theta^{-1}}(P_N)) = q^{-N}$ for all $N \in \mathbb{Z}$. This twisted 2-cocycle descends to the trivial functional on $\mathrm{HH}_2^{\theta^{-1}}(\mathcal{O}(\mathbb{S}_q^2))$.

The other generator of $\mathrm{HC}_{\theta^{-1}}^2(\mathcal{O}(\mathbb{S}_q^2))$, which we denote by $[\phi]$, was recently described by Krähmer [Kr2] and truly corresponds to the “no dimension drop” case in the sense that it is non-trivial on $\mathrm{HH}_2^{\theta^{-1}}(\mathcal{O}(\mathbb{S}_q^2))$. In the following, we compute the pairing of $K_0^{\mathcal{U}_q(\mathrm{su}_2)}(\mathcal{O}(\mathbb{S}_q^2))$ with $[\phi]$.

It has been shown in [Kr2] that $[\phi] = [\varphi + b_2^{\theta^{-1}*} \chi]$, where

$$\varphi(a_0, a_2, a_3) = q\varepsilon(a_0(K^{-1}E \triangleright a_1)(K^{-1}F \triangleright a_1)), \quad a_0, a_1, a_2 \in \mathcal{O}(\mathbb{S}_q^2), \quad (47)$$

and $\chi : \mathcal{O}(\mathbb{S}_q^2) \otimes \mathcal{O}(\mathbb{S}_q^2) \rightarrow \mathbb{C}$ is a linear functional such that

$$(\varphi + b_2^{\theta^{-1}*}\chi)(1, a_1, a_2) = 0 \quad \text{for all } a_1, a_2 \in \mathcal{O}(\mathbb{S}_q^2). \quad (48)$$

Using the vector space basis $\{A^l B^k, A^m B^{n*} : l, k, m \in \mathbb{N}_0, n \in \mathbb{N}\}$ of $\mathcal{O}(\mathbb{S}_q^2)$, χ is determined by

$$\chi(1, A) = (q^{-1} - q)^{-1}, \quad \chi(1, A^2) = (q - q^3)^{-1}, \quad \chi(A, A) = (2(q - q^3))^{-1} \quad (49)$$

and $\chi(A^l B^k, A^m B^{n*}) = 0$ otherwise.

Observe that the evaluation of ϕ on Hochschild cycles reduces to the application of φ from Equation (47). Therefore we will slightly change the Chern character mapping $\text{ch}_2^{\theta^{-1}}$ in Proposition 4.2 in order to obtain Hochschild 2-cycles.

Proposition 5.3. *For the projections P_N defined in Equation (33), set*

$$\overline{\text{ch}}_2^{\theta^{-1}}(P_N) := \text{Tr}_{\theta^{-1}}((1 - 2P_N) \otimes (P_N - \varepsilon(P_N)) \otimes (P_N - \varepsilon(P_N))), \quad (50)$$

where the counit ε of $\mathcal{O}(\text{SU}_q(2))$ is applied to each component of P_N . Then $\overline{\text{ch}}_2^{\theta^{-1}}(P_N)$ is a θ^{-1} -twisted Hochschild 2-cycle, i.e., $b_2^{\theta^{-1}}(\overline{\text{ch}}_2^{\theta^{-1}}(P_N)) = 0$.

Proof. Note that $\varepsilon(P_N)$ is a diagonal complex matrix. In particular, it commutes with $\rho_{|N|/2}^\circ(K^2)$. Using $\text{Tr}_{\theta^{-1}}\varepsilon(P_N)X \otimes Y = \text{Tr}_{\theta^{-1}}X \otimes Y\varepsilon(P_N)$ and $\text{Tr}_{\theta^{-1}}X\varepsilon(P_N) \otimes Y = \text{Tr}_{\theta^{-1}}X \otimes \varepsilon(P_N)Y$ for any $X, Y \in M_{|N|+1 \times |N|+1}(\mathcal{O}(\mathbb{S}_q^2))$, one easily shows that

$$b_2^{\theta^{-1}}(\overline{\text{ch}}_2^{\theta^{-1}}(P_N)) = \text{Tr}_{\theta^{-1}}(1 \otimes (P_N - \varepsilon(P_N))) = 1 \otimes (\text{Tr}_{\theta^{-1}}P_N - \text{Tr}_{\theta^{-1}}\varepsilon(P_N)). \quad (51)$$

Thus it remains to show that $\text{Tr}_{\theta^{-1}}P_N = \text{Tr}_{\theta^{-1}}\varepsilon(P_N)$. Let $l := N/2$. Then

$$\begin{aligned} F \triangleright \text{Tr}_{\theta^{-1}}P_N &= \sum_{k=-|l|}^{|l|} q^{-2k} ((F \triangleright t_{lk}^{|l|*})(K \triangleright t_{lk}^{|l|}) + (K^{-1} \triangleright t_{lk}^{|l|*})(F \triangleright t_{lk}^{|l|})) \\ &= \sum_{k=-|l|}^{|l|} -q^{-(k+1)} \alpha_k^l t_{l,k+1}^{|l|*} t_{lk}^{|l|} + q^{-k} \alpha_{k-1}^l t_{l,k}^{|l|*} t_{l,k-1}^{|l|} = 0 \end{aligned}$$

by Equations (1) and (10). Similarly, $K \triangleright \text{Tr}_{\theta^{-1}}P_N = \text{Tr}_{\theta^{-1}}P_N$, hence $\text{Tr}_{\theta^{-1}}P_N$ belongs to the spin 0 representation of $M_0 = \mathcal{O}(\mathbb{S}_q^2)$. Therefore there exists an $\alpha \in \mathbb{C}$ such that $\text{Tr}_{\theta^{-1}}P_N = \alpha t_{00}^0$. Since $t_{00}^0 = 1$, we can write (with a slight abuse of notation) $\text{Tr}_{\theta^{-1}}\varepsilon(P_N) = \varepsilon(\text{Tr}_{\theta^{-1}}P_N) = \alpha = \text{Tr}_{\theta^{-1}}P_N$ which concludes the proof. \square

Remark 5.4. Actually, $\overline{\text{ch}}_2^{\theta^{-1}}$ and its $2n$ -dimensional analogs give rise to a twisted Chern character from equivariant K_0 -theory into twisted cyclic homology if we replace the (b, B) -bicomplex (38) by the so-called reduced (b, B) -bicomplex. The reduced (b, B) -bicomplex is defined analogous to the (b, B) -bicomplex (38) but with $C_n := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes(n)}$, where $\bar{\mathcal{A}} := \mathcal{A}/\mathbb{C}$, see [Lod] and [GFV] for details in the “non-twisted” case.

We turn now to the computation of the Chern-Connes pairing in the “no dimension drop” case.

Proposition 5.5. *The pairing of $[\phi]$ with $K_0^{\mathcal{U}_q(\text{su}_2)}(\mathcal{O}(S_q^2))$ yields*

$$\langle [\phi], [P_N] \rangle = [N]_q, \quad N \in \mathbb{Z}.$$

Proof. We first claim that $\phi(\text{ch}_2^{\theta^{-1}}(P_N)) = \phi(\overline{\text{ch}}_2^{\theta^{-1}}(P_N))$. By Equation (48), $\phi(1, a, b) = 0$ for all $a, b \in \mathcal{O}(S_q^2)$. Since $\varepsilon(P_N) \in M_{|N|+1 \times |N|+1}(\mathbb{C})$, it suffices to show that $\phi(a, 1, b) = \phi(a, b, 1) = 0$ for all $a, b \in \mathcal{O}(S_q^2)$. For φ , this follows from $E \triangleright 1 = F \triangleright 1 = 0$. Observe that, by (49), $\chi(x, 1) = 0$ for all $x \in \mathcal{O}(S_q^2)$. Hence $b_2^{\theta^{-1}*} \chi(a, 1, b) = \chi(\theta^{-1}(b)a, 1) = 0$ and $b_2^{\theta^{-1}*} \chi(a, b, 1) = \chi(ab, 1) = 0$ which proves the claim.

Since $b_2^{\theta^{-1}*}(\overline{\text{ch}}_2^{\theta^{-1}}(P_N)) = 0$, it follows that $\langle [\phi], [P_N] \rangle = \varphi(\overline{\text{ch}}_2^{\theta^{-1}}(P_N))$. Set $l := |N|/2$ and $n := N/2$. Inserting (33) and (50) into (47) gives

$$\langle [\phi], [P_N] \rangle = q \text{Tr } \sigma_l(K^{-2}) \varepsilon(1 - 2\mathbf{t}_n^{l*} \mathbf{t}_n^l) \varepsilon(K^{-1}E \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l) \varepsilon(K^{-1}F \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l).$$

By (1) and (31), $K^{-1}E \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l = \sigma_l(-qEK) \mathbf{t}_n^{l*} \mathbf{t}_n^l + \sigma_l(K^2) \mathbf{t}_n^{l*} \mathbf{t}_n^l \sigma_l(K^{-1}E)$ and $K^{-1}F \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l = \sigma_l(-q^{-1}FK) \mathbf{t}_n^{l*} \mathbf{t}_n^l + \sigma_l(K^2) \mathbf{t}_n^{l*} \mathbf{t}_n^l \sigma_l(K^{-1}F)$. The complex matrices $\sigma_l(f)$, $f \in \mathcal{U}_q(\text{su}_2)$, can be derived from the formulas in (10). Inserting $\varepsilon(\mathbf{t}_n^{l*} \mathbf{t}_n^l) = (\delta_{ni} \delta_{nj})_{i,j=-l}^l$ yields $\sigma_l(K^{-2}) \varepsilon(1 - 2\mathbf{t}_n^{l*} \mathbf{t}_n^l) = ((-1)^{\delta_{nj}} q^{-2j} \delta_{ij})_{i,j=-l}^l$. If $N \geq 0$, then $n = l$, thus $\varepsilon(K^{-1}E \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l) = (q^l [2l]_q^{1/2} \delta_{li} \delta_{l-1,j})_{i,j=-l}^l$ and $\varepsilon(K^{-1}F \triangleright \mathbf{t}_n^{l*} \mathbf{t}_n^l) = (-q^{l-1} [2l]_q^{1/2} \delta_{l-1,i} \delta_{lj})_{i,j=-l}^l$ by the above. Multiplying the matrices and taking the trace gives $\langle [\phi], [P_N] \rangle = [2l]_q = [N]_q$. A similar calculation shows that $\langle [\phi], [P_N] \rangle = -[2l]_q = [N]_q$ for $N < 0$. \square

6 Orientation

In this section, we show that there exists a twisted Hochschild 2-cycle η in $\mathcal{O}(S_q^2)^{\otimes 3}$ such that $\pi_D(\eta) = \gamma_q$, where

$$\pi_D(\sum_j a_j^0 \otimes a_j^1 \otimes a_j^2) := \sum_j a_j^0 [D, a_j^1] [D, a_j^2] \quad \text{and} \quad \gamma_q := \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}.$$

In analogy to axiom (4') in [C3], we call η a choice of orientation.

In the commutative case, the Hochschild cycle corresponds to the volume form which is non-trivial in de Rham cohomology. For this reason, we consider again the inverse modular automorphism θ^{-1} which avoids the dimension drop.

Proposition 6.1. *With P_1 given in Equation (33) for $N = 1$, define*

$$\eta := \overline{\text{ch}}_2^{\theta^{-1}}(P_1) = \text{Tr}_{\theta^{-1}}((1 - 2P_1) \otimes (P_1 - \varepsilon(P_1)) \otimes (P_1 - \varepsilon(P_1))).$$

Then $b_2^{\theta^{-1}}(\eta) = 0$, the class of η in $\text{HH}_2^{\theta^{-1}}(\mathcal{O}(S_q^2))$ and $\text{HC}_2^{\theta^{-1}}(\mathcal{O}(S_q^2))$ is non-zero,

$$\text{HH}_2^{\theta^{-1}}(\mathcal{O}(S_q^2)) \cong \mathbb{C}[\eta], \quad \text{HC}_2^{\theta^{-1}}(\mathcal{O}(S_q^2)) \cong \mathbb{C}[1] \oplus \mathbb{C}[\eta],$$

and $\pi_D(\eta) = \gamma_q$.

Proof. Since $\eta = \overline{\text{ch}}_2^{\theta^{-1}}(P_1)$, it follows immediately from Proposition 5.3 that η is a θ^{-1} -twisted Hochschild 2-cycle. For brevity of notation, set $n := 1/2$. By (20) and (33),

$$\pi_D(\eta) = \begin{pmatrix} \text{Tr}_{\theta^{-1}}(1 - 2\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_E(\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_F(\mathbf{t}_n^{n*} \mathbf{t}_n^n) & 0 \\ 0 & \text{Tr}_{\theta^{-1}}(1 - 2\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_F(\mathbf{t}_n^{n*} \mathbf{t}_n^n) \partial_E(\mathbf{t}_n^{n*} \mathbf{t}_n^n) \end{pmatrix}.$$

From Equation (46) and the argument following it, we conclude that

$$\pi_D(\eta) = \begin{pmatrix} \text{Tr} \rho_n^\circ(K^2) \mathbf{t}_n^{n*} \mathbf{t}_n^n & 0 \\ 0 & -\text{Tr} \rho_n^\circ(K^2) \mathbf{t}_{-n}^{n*} \mathbf{t}_{-n}^n \end{pmatrix}.$$

Straightforward computations using the embedding (17) and the relations in $\mathcal{O}(\text{SU}_q(2))$ (cf. [KS]) show that

$$\mathbf{t}_n^{n*} \mathbf{t}_n^n = \begin{pmatrix} A & B^* \\ B & 1 - q^2 A \end{pmatrix}, \quad \mathbf{t}_{-n}^{n*} \mathbf{t}_{-n}^n = \begin{pmatrix} 1 - A & -B^* \\ -B & q^2 A \end{pmatrix}, \quad \rho_n^\circ(K^2) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Inserting these matrices into the previous equation and taking the trace gives $\pi_D(\eta) = \gamma_q$.

Observe that

$$\begin{aligned} \eta &= \text{Tr} \rho_n^\circ(K^2) (1 - 2\mathbf{t}_n^{n*} \mathbf{t}_n^n) \otimes (\mathbf{t}_n^{n*} \mathbf{t}_n^n - \varepsilon(\mathbf{t}_n^{n*} \mathbf{t}_n^n)) \otimes (\mathbf{t}_n^{n*} \mathbf{t}_n^n - \varepsilon(\mathbf{t}_n^{n*} \mathbf{t}_n^n)) \quad (52) \\ &= q(1 - 2A) \otimes (A \otimes A + B^* \otimes B) - 2qB^* \otimes (B \otimes A - q^2 A \otimes B) \\ &\quad - 2q^{-1}B \otimes (A \otimes B^* - q^2 B^* \otimes A) + q^{-1}(2q^2 A - 1) \otimes (B \otimes B^* + q^4 A \otimes A). \end{aligned}$$

Let ω_2 denote the θ^{-1} -twisted 2-cycle defined in [Had, Equation (27)]. Then

$$\omega_2 - q^{-1}\eta = 2(B \otimes B^* \otimes A - q^{-2}B \otimes A \otimes B^* - q^2 B^* \otimes A \otimes B + B^* \otimes B \otimes A).$$

Setting

$$\begin{aligned}\beta &= q^{-2}1 \otimes B \otimes B^* \otimes A - q^2 1 \otimes B^* \otimes B \otimes A + q^{-2}1 \otimes A \otimes B \otimes B^* \\ &\quad - q^2 1 \otimes A \otimes B^* \otimes B + q^4 1 \otimes B^* \otimes A \otimes B - q^{-4}1 \otimes B \otimes A \otimes B^*\end{aligned}$$

one easily checks that $b_3^{\theta-1}(2(q^{-2} - q^2)^{-1}\beta) = \omega_2 - q^{-1}\eta$, hence $[\eta] = [q\omega_2]$ in $\mathrm{HH}_2^{\theta-1}(\mathcal{O}(\mathrm{S}_q^2))$ and $\mathrm{HC}_2^{\theta-1}(\mathcal{O}(\mathrm{S}_q^2))$. The last statements of Proposition 6.1 follow now from the results in [Had]. \square

The next proposition shows that η represents the volume form of the covariant differential calculus on $\mathcal{O}(\mathrm{S}_q^2)$.

Proposition 6.2. *For $\sum_j a_j^0 \otimes a_j^1 \otimes a_j^2 \in \mathcal{O}(\mathrm{S}_q^2)^{\otimes 3}$, let*

$$\pi_\wedge(\sum_j a_j^0 \otimes a_j^1 \otimes a_j^2) := \sum_j a_j^0 da_j^1 \wedge da_j^2.$$

Then $\pi_\wedge(\eta) = 2\omega$, where ω denotes the invariant 2-form of Proposition 2.3.

Proof. Again let $n := 1/2$, and set $\tilde{P}_1 := \mathbf{t}_n^{n*} \mathbf{t}_n^n$. From (31), it follows that $f \triangleright \tilde{P}_1 = \sigma_n(S(f_{(1)}))\tilde{P}_1\sigma_n(f_{(2)})$ for $f \in \mathcal{U}_q(\mathrm{su}_2)$. By (32), $\rho_n^\circ(K^2) = \sigma_n(K^{-2})$. Thus, with η given by (52),

$$\begin{aligned}f \triangleright \pi_\wedge(\eta) &= \mathrm{Tr} \sigma_n(K^{-2})\sigma_n(S(f_{(1)}))(1 - 2\tilde{P}_1)d\tilde{P}_1 \wedge d\tilde{P}_1\sigma_n(f_{(2)}) \\ &= \mathrm{Tr} \sigma_n(f_{(2)}K^{-2}S(f_{(1)}))(1 - 2\tilde{P}_1)d\tilde{P}_1 \wedge d\tilde{P}_1 = \varepsilon(f) \pi_\wedge(\eta),\end{aligned}$$

where we used the cyclicity of the trace and $K^2 f K^{-2} = S^2(f)$ for $f \in \mathcal{U}_q(\mathrm{su}_2)$. Hence $\pi_\wedge(\eta)$ is an invariant 2-form. By Proposition 2.3, ω is also invariant and $\Omega^{\wedge 2} \cong \mathcal{O}(\mathrm{S}_q^2)\omega$. Therefore $\pi_\wedge(\eta) = c\omega$, where $c = \int \pi_\wedge(\eta)$ and, by (22),

$$\int \pi_\wedge(\eta) = h(q \mathrm{Tr}_{\theta^{-1}}(1 - 2\tilde{P}_1)\partial_E(\tilde{P}_1)\partial_F(\tilde{P}_1) - q^{-1} \mathrm{Tr}_{\theta^{-1}}(1 - 2\tilde{P}_1)\partial_F(\tilde{P}_1)\partial_E(\tilde{P}_1)).$$

The proof of Proposition 6.1 shows that $\mathrm{Tr}_{\theta^{-1}}(1 - 2\tilde{P}_1)\partial_F(\tilde{P}_1)\partial_E(\tilde{P}_1) = -q$ and $\mathrm{Tr}_{\theta^{-1}}(1 - 2\tilde{P}_1)\partial_E(\tilde{P}_1)\partial_F(\tilde{P}_1) = q^{-1}$. Hence $c = 2$. \square

7 Index computation and Poincaré duality

The K -theoretic version of Poincaré duality states that the additive pairing on $K_*(\mathcal{C}^*(\mathrm{S}_{qs}^2))$ determined by the index map of D is non-degenerate [C2, GFV]. Since $K_1(\mathcal{C}^*(\mathrm{S}_{qs}^2)) = 0$ [MNW], it suffices, in our case, to verify the non-degeneracy of the pairing $\langle \cdot, \cdot \rangle_D : K_0(\mathcal{C}^*(\mathrm{S}_{qs}^2)) \times K_0(\mathcal{C}^*(\mathrm{S}_{qs}^2)) \rightarrow \mathbb{Z}$,

$$\langle [P], [Q] \rangle_D := \mathrm{ind}((P \otimes JQJ^*)\bar{\partial}_F(P \otimes JQJ^*)),$$

where $P \in M_{n \times n}(\mathcal{C}^*(S_{qs}^2))$ and $Q \in M_{m \times m}(\mathcal{C}^*(S_{qs}^2))$ are projections, $\bar{\partial}_F$ denotes the lower-left entry of D , and $(P \otimes JQJ^*)\bar{\partial}_F(P \otimes JQJ^*)$ is an unbounded Fredholm operator mapping from its domain in $(P \otimes JQJ^*)\overline{M}_{-1}^{n \times m}$ into the Hilbert space $(P \otimes JQJ^*)\overline{M}_1^{n \times m}$.

Recall that an unbounded Fredholm operator F is an operator between Hilbert spaces with dense domain, finite-dimensional kernel, and finite-codimensional range. Its index $\text{ind}(F)$ is the difference between the dimensions of kernel and cokernel or, equivalently, $\text{ind}(F) = \dim(\ker F) - \dim(\ker F^*)$. For an $\mathcal{U}_q(\text{su}_2)$ -equivariant Fredholm operator F (i.e., F commutes with the action of $\mathcal{U}_q(\text{su}_2)$ on the Hilbert space), kernel and cokernel carry a finite-dimensional representation of $\mathcal{U}_q(\text{su}_2)$ and we define

$$q\text{-ind}(F) := \text{Tr}_{\ker F} K^2 - \text{Tr}_{\ker F^*} K^2.$$

The next lemma is the key to index computations of D .

Lemma 7.1. *For $N \in \mathbb{Z}$, let $\bar{\partial}_E^N$ and $\bar{\partial}_F^N$ denote the closures of the operators $\partial_E : M_N \rightarrow \overline{M}_{N-2}$ and $\partial_F : M_N \rightarrow \overline{M}_{N+2}$, respectively, given by (11). Then*

$$\begin{aligned} \ker(\bar{\partial}_E^N) &= \{0\}, \quad \ker(\bar{\partial}_F^N) = \text{span}\{v_{N/2,j}^{N/2} : j = -\frac{N}{2}, \dots, \frac{N}{2}\}, \quad N > 0, \\ \ker(\bar{\partial}_E^N) &= \text{span}\{v_{N/2,j}^{|N|/2} : j = -\frac{|N|}{2}, \dots, \frac{|N|}{2}\}, \quad \ker(\bar{\partial}_F^N) = \{0\}, \quad N < 0, \end{aligned}$$

and $\ker(\bar{\partial}_E^0) = \ker(\bar{\partial}_F^0) = \mathbb{C}v_{00}^0$.

Proof. We give the proof for $\bar{\partial}_F^N$, the other case is similar. By (11), $v_{N/2,j}^l$ is an eigenvector of $|\bar{\partial}_F^N|$ with corresponding eigenvalue $\alpha_{N/2}^l$. The claim follows from $\ker(\bar{\partial}_F^N) = \ker(|\bar{\partial}_F^N|)$ since $\alpha_{N/2}^l = 0$ if and only if $N \geq 0$ and $l = N/2$. \square

We turn now to an explicit computation of indices.

Proposition 7.2. *For $N \in \mathbb{Z}$,*

$$\text{ind}(P_N \bar{\partial}_F P_N) = N, \quad q\text{-ind}(P_N \bar{\partial}_F P_N) = [N]_q.$$

Remark 7.3. Note that $\langle [\frac{1}{2}\tau], [P_N] \rangle = q\text{-ind}(P_N \bar{\partial}_F P_N) = \langle [\phi], [P_N] \rangle$. The first equality already been established in [NT1] from a local index formula. Whether $\langle [\phi], [P_N] \rangle$ can be computed from a local index formula will be discussed elsewhere.

Proof. Let $n := N/2 > 0$ and set $\tilde{P}_N := \mathbf{t}_n^{n*} \mathbf{t}_n^n$. By the definition of P_N in (33), $\ker(P_N \bar{\partial}_F P_N) = \rho_n^\circ(K)^{-1} \ker(\tilde{P}_N \bar{\partial}_F \tilde{P}_N)$. Suppose we are given an

$\mathbf{x}_{-1} \in \overline{M}_{-1}^{2n+1}$ such that $\tilde{P}_N \bar{\partial}_F \tilde{P}_N \mathbf{x}_{-1} = 0$. From $\partial_F(\mathbf{t}_n^*) \sim \mathbf{t}_{n-1}^{n*}$, $\mathbf{t}_n^n \mathbf{t}_{n-1}^{n*} = 0$, $\mathbf{t}_n^n \mathbf{t}_n^{n*} = 1$ and Equation (12), we conclude that $\tilde{P}_N \bar{\partial}_F \tilde{P}_N \mathbf{x}_{-1} = 0$ if and only if $\mathbf{t}_n^{n*} \bar{\partial}_F^{N-1}(\mathbf{t}_n^n \mathbf{x}_{-1}) = 0$. From [SW3, Lemma 6.5(iii)], it follows that $\ker(t_{n,-n}^{n*}) = 0$. Thus $\mathbf{t}_n^{n*} \bar{\partial}_F^{N-1}(\mathbf{t}_n^n \mathbf{x}_{-1}) = 0$ if and only if $\bar{\partial}_F^{N-1}(\mathbf{t}_n^n \mathbf{x}_{-1}) = 0$ since $\mathbf{t}_n^{n*} = (t_{n,-n}^{n*}, \dots, t_{n,n}^{n*})^t$. As $\mathbf{t}_n^n \mathbf{x}_{-1} \in \overline{M}_{N-1}$, Lemma 7.1 shows that

$$\ker(\tilde{P}_N \bar{\partial}_F \tilde{P}_N) = \{\mathbf{x}_{-1} \in \overline{M}_{-1}^{2n+1} : \mathbf{t}_n^n \mathbf{x}_{-1} \in \text{span}\{v_{n-1,-n+1}^{n-1}, \dots, v_{n-1,n-1}^{n-1}\}\}.$$

By Lemma 2.1, we can find $\mathbf{x}_{-1}^j \in M_{-1}^{2n+1}$ such that $\mathbf{t}_n^n \mathbf{x}_{-1}^j = v_{n-1,j}^{n-1}$, $j = -(n-1), \dots, n-1$. Set $\tilde{\mathbf{x}}_{-1}^j := \rho_n^\circ(K)^{-1}(\mathbf{x}_{-1}^j)$. By the preceding, $\tilde{\mathbf{x}}_{-1}^j \in \ker(P_N \bar{\partial}_F P_N)$. Suppose that $\tilde{\mathbf{y}}_{-1}^{-n}, \dots, \tilde{\mathbf{y}}_{-1}^n$ is another such set. Then

$$P_N(\tilde{\mathbf{x}}_{-1}^j - \tilde{\mathbf{y}}_{-1}^j) = \rho_n^\circ(K^{-1}) \mathbf{t}_n^{n*} \mathbf{t}_n^n (\mathbf{x}_{-1}^j - \mathbf{y}_{-1}^j) = \rho_n^\circ(K^{-1}) \mathbf{t}_n^{n*} (v_{n-1,j}^{n-1} - v_{n-1,j}^{n-1}) = 0,$$

so $P_N \tilde{\mathbf{x}}_{-1}^j = P_N \tilde{\mathbf{y}}_{-1}^j$ in $P_N \overline{M}_{-1}^{2n+1}$. Hence

$$\ker(P_N \bar{\partial}_F P_N) \cong \text{span}\{v_{n-1,-(n-1)}^{n-1}, \dots, v_{n-1,n-1}^{n-1}\}. \quad (53)$$

Now we consider $(P_N \bar{\partial}_F P_N)^* = P_N \bar{\partial}_E P_N$. Using $\partial_E(\mathbf{t}_n^*) = 0$, the same reasoning as above shows that $\tilde{P}_N \bar{\partial}_E \tilde{P}_N \mathbf{x}_1 = 0$ for $\mathbf{x}_1 \in \overline{M}_1^{2n+1}$ if and only if $\bar{\partial}_E(\mathbf{t}_n^n \mathbf{x}_1) = \bar{\partial}_E^{N+1}(\mathbf{t}_n^n \mathbf{x}_1) = 0$. Lemma 7.1 implies $\mathbf{t}_n^n \mathbf{x}_1 = 0$, thus $P_N(\rho_n^\circ(K)^{-1} \mathbf{x}_1) = 0$ and therefore $\ker(P_N \bar{\partial}_E P_N) = \{0\}$. Summarizing, we get

$$\begin{aligned} \text{ind}(P_N \bar{\partial}_F P_N) &= \dim(\ker(P_N \bar{\partial}_F P_N)) = 2(n-1) + 1 = N, \\ q\text{-ind}(P_N \bar{\partial}_F P_N) &= \text{Tr}_{\ker(P_N \bar{\partial}_F P_N)} K^2 = \sum_{j=-n+1}^{n-1} q^{2j} = [N]_q \end{aligned}$$

by (53). This proves the proposition for $N > 0$. The case $N = 0$ is trivial, and the case $N < 0$ is proved similarly with the role of $\bar{\partial}_F$ and $\bar{\partial}_E$ interchanged. \square

It has been shown in [MNW] that $\mathcal{C}^*(S_{qs}^2) \cong \mathbb{C}1 + \mathcal{K}(\ell^2(\mathbb{N}_0))$, where $\mathcal{K}(\ell^2(\mathbb{N}_0))$ denotes the compact operators on $\ell^2(\mathbb{N}_0)$, and $K_0(\mathcal{C}^*(S_{qs}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$. The generator $[(1, 0)]$ was taken to be the identity $P_0 = 1$, and the other generator $[(0, 1)]$ the 1-dimensional projection onto the first basis vector. From [MNW] and [Haj], we conclude that $[P_1] = [(1, 1)]$. In particular, $[P_0]$ and $[P_1]$ also generate $K_0(\mathcal{C}^*(S_{qs}^2))$.

Using the results from the previous proposition, we can easily establish Poincaré duality:

Proposition 7.4. *The pairing $\langle \cdot, \cdot \rangle_D : K_0(\mathcal{C}^*(S_{qs}^2)) \times K_0(\mathcal{C}^*(S_{qs}^2)) \rightarrow \mathbb{Z}$ is given by*

$$\langle [(k, l)], [(m, n)] \rangle_D = km - ln, \quad k, l, m, n \in \mathbb{Z}.$$

In particular, it is non-degenerate, so Poincaré duality holds.

Proof. We first show that $\langle \cdot, \cdot \rangle_D$ is antisymmetric. Let P and Q be projection matrices with entries in $\mathcal{C}^*(S_{qs}^2)$. Note that $\text{ind}((P \otimes Q) \bar{\partial}_F (P \otimes Q)) = \text{ind}((Q \otimes P) \bar{\partial}_F (Q \otimes P))$ since the flip of tensor factors is an unitary operation which commutes with the component wise action of $\bar{\partial}_F$. Next, $J^* P J = J P J^*$ since $J^2 = -1$. Recall from Section 2.3 that D and J are odd operators, i.e., $\gamma D = -D \gamma$ and $\gamma J = -J \gamma$, and $J D = D J$. Hence $J^* \bar{\partial}_F J = \bar{\partial}_E = \bar{\partial}_F^*$. Moreover, $\text{ind}(F) = -\text{ind}(F^*)$ for any Fredholm operator F . Thus

$$\begin{aligned} \langle [P], [Q] \rangle_D &= \text{ind}((P \otimes J Q J^*) \bar{\partial}_F (P \otimes J Q J^*)) \\ &= \text{ind}(J(J^* P J \otimes Q) J^* \bar{\partial}_F J(J^* P J \otimes Q) J^*) \\ &= -\text{ind}((Q \otimes J P J^*) \bar{\partial}_F (Q \otimes J P J^*)) = -\langle [Q], [P] \rangle_D. \end{aligned}$$

By the additivity of the pairing, it suffices to consider the generators $[P_1] = [(1, 1)]$ and $[P_0] = [(1, 0)]$. From Proposition 7.2, we get immediately $\langle [(1, 1)], [(1, 0)] \rangle_D = 1$, and the antisymmetry implies $\langle [(1, 0)], [(1, 1)] \rangle_D = -1$, $\langle [(1, 0)], [(1, 0)] \rangle_D = \langle [(1, 1)], [(1, 1)] \rangle_D = 0$. For $k, l, m, n \in \mathbb{Z}$, we obtain now

$$\begin{aligned} \langle [(k, l)], [(m, n)] \rangle_D &= \langle k[(1, 1)] + (l-k)[(1, 0)], m[(1, 1)] + (n-m)[(1, 0)] \rangle_D \\ &= kn - lm, \end{aligned}$$

which completes the proof. \square

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References

- [CPR] Carey A. L., J. Phillips and A. Rennie: *Twisted cyclic theory and an index theory for the gauge invariant KMS state on Cuntz algebras*. Preprint, arXiv:0801.4605v2 [math.KT].
- [CP] Chakraborty, P. S. and A. Pal: *Equivariant spectral triples on the quantum SU(2) group*, *K-Theory* **28** (2003), 107–126.
- [C1] Connes, A.: *Noncommutative geometry*. Academic Press, San Diego, 1994.

- [C2] Connes, A.: *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995), 6194–6231.
- [C3] Connes, A.: *Gravity coupled with matter and the foundation of non-commutative geometry*, Commun. Math. Phys. **182** (1996), 155–176.
- [C4] Connes, A.: *Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$* , J. Inst. Math. Jussieu **3** (2004), 17–68.
- [DDLW] Dąbrowski, L., F. D’Andrea, G. Landi and E. Wagner: *Dirac operators on all Podleś quantum spheres*, J. Noncommut. Geom. **1** (2007), 213–239.
- [DLSSV1] Dąbrowski, L., G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly: *The Dirac operator on $SU_q(2)$* , Commun. Math. Phys. **259** (2005), 729–759.
- [DLSSV2] Dąbrowski, L., G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly: *The local index formula for $SU_q(2)$* , K-Theory **35** (2005), 375–394.
- [DS] Dąbrowski, L. and A. Sitarz: *Dirac operator on the standard Podleś quantum sphere*, Banach Centre Publications **61** (2003), 49–58.
- [GFV] Gracia-Bondía, J. M., H. Figueroa and J. C. Várilly: *Elements of Noncommutative Geometry*. Birkhäuser, Boston, 2001.
- [Had] Hadfield, T.: *Twisted cyclic homology of all Podleś quantum spheres*, J. Geom. and Phys. **57** (2007), 339–351.
- [Haj] Hajac, P. M.: *Bundles over quantum sphere and noncommutative index theorem*, K-Theory **21** (1996), 141–150.
- [KS] Klimyk, A. U. and K. Schmüdgen: *Quantum Groups and their Representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
- [Kr1] Krähmer, U.: *Dirac operators on quantum flag manifolds*, Lett. Math. Phys. **67** (2004), 49–59.
- [Kr2] Krähmer, U.: *The Hochschild cohomology ring of the standard Podleś quantum sphere*. Preprint, arXiv:0806.1615v1 [math.QA].
- [KMT] Kustermans J., G. Murphy and L. Tuset: *Differential calculi over quantum groups and twisted cyclic cocycles*, J. Geom. Phys. **44** (2003), 570–594.

- [Lod] Loday, J.-L.: Cyclic homology. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1992.
- [MNW] Masuda, T., Y. Nakagami and J. Watanabe: *Noncommutative differential geometry on the quantum two sphere of Podleś. I: An Algebraic Viewpoint*, *K-Theory* **5** (1991), 151–175.
- [NT1] Neshveyev, S. and L. Tuset: *Hopf algebra equivariant cyclic cohomology, K-theory and index formulas*, *K-Theory* **31** (2004), 357–378.
- [NT2] Neshveyev, S. and L. Tuset: *A Local Index Formula for the Quantum Sphere*, *Commun. Math. Phys.* **254** (2005), 323–341.
- [Pod] Podleś, P.: *Quantum spheres*, *Lett. Math. Phys.* **14** (1987), 193–202.
- [Sch] Schmüdgen, K.: *Commutator representations of differential calculi on the group $SU_q(2)$* , *J. Geom. Phys.* **31** (1999), 241–264.
- [SW1] Schmüdgen, K. and E. Wagner: *Hilbert space representations of cross product algebras*, *J. Funct. Anal.* **200** (2003), 451–493.
- [SW2] Schmüdgen, K. and E. Wagner: *Dirac operator and a twisted cyclic cocycle on the standard Podleś quantum sphere*, *J. Reine Angew. Math.* **574** (2004), 219–235.
- [SW3] Schmüdgen, K. and E. Wagner: *Representations of cross product algebras of Podleś quantum spheres*, *J. Lie Theory* **17** (2007), 751–790.
- [Sit] Sitarz, A.: *Equivariant spectral triples*, *Banach Centre Publications* **61** (2003), 231–263.
- [Wor] Woronowicz, S. L.: *Differential calculus on compact matrix pseudogroups (quantum groups)*, *Commun. Math. Phys.* **122** (1989), 125–170.